ESSAY REVIEW

"MATHEMATICS AS AN EDUCATIONAL TASK" BY H. FREUDENTHAL (DORDRECHT, THE NETHERLANDS: REIDEL, 1975)

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The Background

In these days of change and "progress" in education, we are placed in the position of needing the wisdom of Solomon to decide which of a wide variety of views is appropriate, if indeed any of them are. The wave of "new mathematics" began to sweep the country in the 60's and now in the 70's we are looking back at the improvements and the debris. On the one hand we have advocates of new teaching methods, on the other a traditionalist backbone asks for retention of old arithmetical skills taught by rote. It is not even a simple choice of old or new for there is an embarrassing choice of "new" methods in mathematics teaching. The developmental psychology of Jean Piaget has greatly influenced teaching of mathematics at the primary level with the Nuffield Mathematics Project, for example, taking Piaget's views as axiomatic. Politics have entered the fray with the great debate over comprehensive education. Sociologists, philosophers, curriculum theorists, manufacturers of educational aids, educationists of all shades of opinion have added their voices. One wonders whether mathematical education is an intellectual discipline or a growth industry – all of this against a background where the latest U.K. government figures tell us that there is a shortage of 1200 qualified mathematics teachers in schools.

Literally thrown into the maelstrom like a gauntlet we have the text *Mathematics as an Educational Task* by Hans Freudenthal. It is a monumental work in mathematical education terms, collecting together the ideas which Professor Freudenthal has developed over the years, forming the most comprehensive corpus of thought on the subject ever published in a single volume by one man. For those committed to mathematical education it is an essential book to read. Professor Freudenthal is a considerable force to be reckoned with. He is a leading light in mathematical education in Holland having previously established himself as a research mathematician of the first rank. He is also a perceptive critic who will not flinch from exposing the charlatan or from strongly stating his views when he simply holds a different opinion from another whom he respects. For a long and consistent period he has applied his energies to intense thought about the problems of mathematical instruction investigating the research of others and working with teachers as well as being directly involved with teaching himself. As a result he has come to trust the conviction of experience rather than the outcome of many experimental investigations in what he calls "mathematical-didactical problems", the results of which he regards as being meaningless. This book, in the words of the author is "above all, a philosophy of mathematical education".

The Philosophy

In an illuminating preface the author gives the ethos of his approach to education. For many workers in the subject his views will be found to be startling (although others would give him considerable support):

> "There is no need to embellish low key education using highbrow psychology … Misusing Piaget's name has become quite a habit in the didactical literature."
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> "For a few other reasons I did not mention mathematical-didactical experiments … My criticism is aimed at the spirit behind such research. Embellishing it with a statistical analysis does not mean that the rigour of natural science has been transferred to educational research."
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> "Rather than from such experimental investigations, I learned a lot from my own and from reported classroom experiences, from text books and manuals whether I liked them or not, and from honest analysis of subject matter and learning behaviour as performed by teachers. True educational activity means tracing the right path to education guided by one's honest conviction. Educational science should, first of all, be a rational justification of this honest conviction."

All of these quotations are taken from the preface and I hope I do the author no serious disservice by quoting them out of context, especially the last sentence, which seems to me to contain seeds of potential danger. History has shown what happens when people "rationally justify their honest convictions".

It is, therefore, important when reading the text to be aware that the author is putting forward his own highly personal viewpoint. He does this with compelling eloquence. Professor Freudenthal's native tongue is not English, his style is remarkably fluent but with a continental abrasiveness that contrasts strongly with the more deferential idiomatic usage to be found amongst most British authors. This gives the text a sharpness and a liveliness which often administers an abrupt shock to the reader, provoking thought of an often retaliatory nature as an incisive comment strikes home.

The Contents

Although the text is written in nineteen chapters it breaks down naturally into two distinct parts, the first ten chapters (169 pages) are various discourses on aspects of mathematical education, the remaining nine (507 pages) being devoted to the contents of the mathematical curriculum reviewed in the broadest possible context. There follow two appendices, one being his criticism of aspects of Piaget's work and the other a list of his papers on mathematical instruction.

It is important to mention two sizeable omissions at the outset, namely the lack of adequate references to other works and the lack of any form of index to the book itself. The author justifies the first of these in characteristic fashion:

> "I have avoided all citations with respect to textbooks, designs and lessons wherever it is feasible. I believe I had a compelling reason to do so for this material was frequently subject to criticism which was in fact negative. The material could be sharply divided into serious work and trash. Citations in footnotes would have meant tarring everything with the same brush. This I would hate to do … It would have been too much honour for trash to be quoted alongside serious literature."

Anyone familiar with a tenth of the material available for mathematical instruction today will have some sympathy with Professor Freudenthal's point of view. To criticise or praise some aspect of a particular textbook without checking the validity of others on the same topic would lead to invidious decisions being taken. With the text aimed at an international market the problem is multiplied. Nevertheless omission of references often weakens the text as an academic exercise for the less informed reader. A footnote like the following: "A misinterpretation of this theory is the most spectacular showpiece of modern school texts" without any reference to a source material may be of little help to the reader without experience of the texts concerned. There are many such people, including, for instance, a fair proportion of established university mathematicians who have not studied the contents of new mathematics courses in detail. Professor Freudenthal's decision to omit references of this nature implies that we are sometimes left with his assessment of a situation without the means to make a judgement for ourselves.

The other omission, the index, could be easily remedied in a future edition. It is not a pleasant task to recall some material embedded somewhere in a 700 page book without guidance from an index.

TOPICS ON MATHEMATICAL EDUCATION

The first nine chapters are rich material indeed. There is a ripeness in his language which speaks of a lifetime's experience of mathematics seen as a part of the ever-flowing stream of historical development as a research worker and teacher. Sometimes a chapter will read as though the author set out a title, then with a glass of mature wine in his hand he sat with his pupils at his feet and told of his philosophy. There flows a profusion of anecdotes, *bon mots,* words of wisdom, tumbling one after another in an eminently compulsive and readable style. It leaves one marvelling at the thoughts, sometimes finding one's own ideas, eloquently put into words, other times finding glimpses of ideas as yet not fully distilled in one's own mind being formulated by the author. He begins by briefly reviewing the mathematical tradition, every word speaking volumes about the depths of his insight into history. Later this chapter will allow us to see so-called "modern" mathematics as part of the ever-changing pattern. By seeing the growth and collapse of the importance of various theories in history we are much more able to set ourselves apart and look more objectively at current changes in the curriculum. As, always, in this text however, we see history through the shades of the author's conviction with all the resulting benefits and drawbacks.

Chapter II looks at "Mathematics Today". This is a valuable distillation of current trends wherein the author tells us that it is not so much content but style and approach that have changed over the last few decades. "Variables" have been eclipsed in favour of a set-theoretic approach via functions, mathematics has become more formalised and axiomatised, geometrical intuition has become viewed through algebraic symbolism and so on. Even in this chapter the author begins to mount his attack on the psychology of Piaget using a double-edged sword of irony to cut the formalism of Bourbaki* at the same time.

> "The most spectacular example of organising mathematics is, of course, Bourbaki. How convincing this organisation of mathematics is! So convincing that Piaget could rediscover Bourbaki's system in developmental psychology. Poor Piaget! He did not fare much better than Kant, who had barely consecrated Euclidean space as a "pure intuition" when non-Euclidean geometry was discovered! Piaget is not a mathematician, so he could not know how unreliable mathematical system builders are. Bourbaki's system of mathematics was not yet accomplished when the importance of categories was discovered. There can be little doubt that categories will be a new organising principle and that rebuilding of Bourbaki's structure in categorical style will leave no stone left on top of another. If a leading developmental psychologist could then convince us of the categorizing genesis of all mathematical concepts – which will certainly eventually happen – then it will just be in time to see categorical style mathematics, before it is ready, being pulled down in favour of some new principle which will certainly have its day … If the reader does not know what categories are he should not worry too much since the writer of the present book is not convinced what they are either …"

* Bourbaki is the pseudonym of a group of French mathematicians who set out to organize mathematical knowledge into a coherent logical system.

I have given a more extended quotation this time to show something of the style of the author in full flight. Freudenthal returns several times to lambast Piaget, so much so that he eventually brings together his criticism into the appendix. In the given quotation we also see the barbs striking at the opposite end of the scale – the formalist school who believe in building up mathematics not from within the learner, developmentally á la Piaget, but from within mathematics itself as a separate logical structure divorced from the real world. Freudenthal has much more to say on this score when he looks at the mathematical curriculum later in the text.

In chapter III "Tradition and Education" we again meet the changing face of modern education to a point where change itself becomes the modern tradition. This is one of the chapters which I spoke of earlier when the author gives the benefit of his experience in a variety of ways without seeming to keep a single thread of thought to bind the ideas together. We have numerous anecdotes concerning the way tradition and pre-judgements blinded society until someone dared to doubt the system and proved it lacking. Half way through the chapter the author himself voices the opinion that the reader might think that he has confused the issue, starting with a belief in tradition, then gradually changing this belief to suggest that what really concerns him is the change in style of instruction, from the traditional manner of passive listening to a new way of active acquisition. He sees the latter aim as not being in conflict with the idea of mathematics education as a mass activity with impersonal programmed instruction and ends up with a plea for an image of mathematics for the totality of education. It is as if the author started out with a basic idea in mind but like a starling attracted by bright objects, he alighted on many new things and being loth to delete his material and start again more clearly organized, he decided to leave his work as it came. In some ways there is no harm in this, for as I look back over the chapter in my copy of the book I find I have pencilled many comments in the margin and underlined a number of Freudenthal's remarks with notes of approval. In one of his anecdotes on the perils of good teaching he says

> "I saw demonstration lessons in mathematics … where teachers of the highest quality lectured unswervingly for three quarters of an hour to an audience which did not dare swallow".

Perhaps if everything is too well-organised the audience will learn nothing. If we read this chapter in a spirit of active acquisition there is a quality in its wayward anecdotes which can provoke useful thinking.

In chapter IV we come to "The Use and Aim of Mathematical Instruction". The author has little time for laudable aims in mathematical education. "In no other field of instruction is the distance between useless aim and aimless use so great". Here he launches an attack on experiments which investigate the teaching of isolated pieces of mathematics; such a success, in his words "is a cheap success, because any isolated matter can be taught successfully if it is done forcefully". He also attacks one of the basic tenets of our primary system, that one of the aims of teaching mathematics should be to make it enjoyable. "This is a hypocritical argument because we are not used to choosing subjects according to the criterion of student's pleasure ... I do not claim that the worse food tastes the more healthy it is. I simply say that taste, too, should be educated".

He has some most interesting remarks to make with regard to the applications of mathematics but makes his main theme that mathematics should be "fraught with relations". This quaint phrase refers to the need for the curriculum to weld strong linkages between concepts making a coherent schema. For instance he does not advocate learning mathematics for isolated applications because such mathematics soon fossilises. What he does advocate is to learn how to apply mathematics and coins the phrase "multi-related" to back up his notions. In a manner with which we are now familiar he goes on to discuss many other topics — mathematics as a "discipline of the mind", as a "means of selection" and a "language" and so on.

Now we turn to teaching methods. The next chapter on "The Socratic Method" is a model of how Freudenthal's style can be a positive asset. Ostensibly the title is about the manner of teaching in which the teacher pre-thinks the ideas from an imaginary pupil's viewpoint, then presents the material formally as a discourse between the pupil and himself. Freudenthal advocates this method for formal lectures with the lecturer playing both parts, as his own devil's advocate so to speak. He uses this basis as a springboard from which to look at other teaching styles; ten pages of densely packed and highly recommended thought.

In chapter VI Freudenthal looks at "Re-invention" in terms of activity mathematics as opposed to the learning of ready-made material. As we have seen in the phrase "mathematics fraught with relations", he is very keen to see that the ideas in mathematics become closely integrated into a coherent whole in the mind of the learner, so it is no surprise to see that he says:

> "The best way to learn an activity is to perform it."

However, he has a lot of criticisms to hand out, to Piaget (p. 120), to Dienes (p. 127), and others, whilst he gives praise to the Dutch research worker, Dina van Hiele, for her analysis of the levels in the learning process. The criticism of Piaget we will deal with later when we consider the appendix but the work of Dienes must be defended now. Freudenthal gives praise to the Van Hieles for work which demonstrates that concept formation goes through several clearly defined stages and criticises Dienes on the grounds that his research seems rooted in the bottom (pre-mathematical) stage.

Clearly in his book *Building up Mathematics* (1960) Dienes was aware of such stages because he describes his own version incorporating six stages in chapter 2. Whether he was aware of Van Hiele's research incorporated in *Report on Methods of Initiation into Geometry* (ed. Freudenthal, 1958) is another question not discussed by Freudenthal. Certainly Dienes makes no reference to it in his book and it is unlikely that he would have omitted such a reference were it instrumental in his thinking. The point at issue is simply that Dienes was well aware of the spectrum of stages when doing his research and did not limit himself to the lowest one.

Exciting possibilities are in the air when chapter VII looks at "Organisation of a Field by Mathematizing" but, to retain an earlier metaphor, the wine proves to be too strong. After the content of earlier chapters we might hope to read about the virtues of mathematical modelling and the link between the real world and abstract mathematics. It begins promisingly enough but strays into the problems of aligning the curriculum for mathematics in schools with that of physics. A few words on algorithms for general use and its complementary counterpart, the singular example of an isolated phenomenon such as the irrationality of $\sqrt{2}$ or the existence of an infinite number of primes, then the chapter dies inexplicably away.

The mental fibre returns in chapter VIII on "Mathematical Rigour" where the author points out that there are various levels of rigour appropriate under different circumstances. This needs saying and it needs to be said by a good practising mathematician. Professor Freudenthal does the honours.

> "It is blindness to assert there is only one rigour (which of course happens to be that exercised by the person who is asserting it) and any others below are false and all others above are hair splitting".

Chapters IX and X complete the first part of the book on topics in mathematical education being obligatory considerations of "instruction" and "the mathematics teacher". Each of these is short, representing the position of the author as a university professor being well acquainted with university teaching but more of an onlooker at the school level. This viewpoint has its value, however, for instance, his position lends more authority to his realisation that it is not the job of university professors to prescribe the methods or content of school courses.

> "I do not believe it is the essence of this book and of what I used to advocate in teaching theory that I should prescribe to practising educators their teaching methods. But it certainly has been, and still is, my philosophy to fight any attempts at influencing school instruction in any anti-didactical way. In particular I oppose all purely content-oriented instruction, and all dogmatic views on mathematics which neglect all the psychological presuppositions and social implications of mathematics

> instruction. Such dogmatic efforts are often made by university professors who by their very nature are prone to view school mathematics as subject matter and as a system".

Freudenthal looks at the problems of teaching and the teacher within the area of his experience in Holland although a number of his conclusions have wider relevance. He sees the individualised instruction at university as an anachronism and looks forward to the introduction of a type of programmed learning which encourages active acquisition and re-invention. He also looks forward to more experience of team teaching with teachers learning from one another. As in England, he sees the quality of teachers as a major consideration in Holland and formulates minimum demands for the training of mathematics teachers, focussing in particular on the need for training and retraining primary school teachers. These demands involve mainly knowledge and understanding of mathematics in a self-reliant way, to see how mathematics is applied and to have an insight into how mathematicians do research (for secondary teachers). In view of his critical destruction of Piaget and other psychologists, it is interesting to see how in the above quotation, he can oppose the neglect of all psychological presuppositions. He supplies his answer by saying

> "If we urge today that theory should be re-invented (and therefore cannot precede practice) there is only a small amount of teaching theory left to be learned before a teacher starts teaching, namely to learn from one's own and others' example to analyse the instruction one is attempting to give, is giving, and has been giving".

In the remaining nine chapters this is what he proceeds to do.

THE CURRICULUM

The major portion of the text is spent on a study of the mathematics curriculum. Four chapters see the number concept from various points of view, including its extension to algebraic notions, then a chapter each on set theory, geometry, analysis, probability and statistics, and logic.

This is curriculum study in the widest sense involving not only the subject matter but also its interpretation at various levels, its suitability, its application, its usefulness, and so on. Freudenthal is on firmer ground here and the contents of his text are easier to formulate. This is because the mathematics itself has an underlying structure. Not only does this mean that the analysis of the content can be seen more clearly to be right or wrong within the logical paradigm of mathematics but also the structure itself often imposes some sort of order on the discussion, giving the chapters themselves a clearer overall development.

As an example let us look at Freudenthal's criticism of the number concept as viewed by Piaget. In chapter XI he considers the cardinal aspect of number which Piaget regards as fundamental, namely two (finite) sets have the same number of elements if they can be placed in one-one correspondence. This Freudenthal calls the 'numerosity number' as opposed to the "counting number". Thus a set of cups and a set of saucers have the same numerosity number if each cup is placed on a separate saucer with all the cups and saucers utilised in this way. This does not involve counting. To use number in a counting sense one must count the cups then the saucers and check that the result is the same. It is Freudenthal's contention that counting is so efficient it soon replaces the numerosity aspect with children, though Piaget and set theoreticians alike overstress the latter. He considers the numerosity number and shows it to be mathematically insufficient and unimportant. His insufficiency criterion is that numerosity number of finite sets cannot suffice as a foundation of the natural numbers. Simply put, the set of all natural numbers is itself an infinite set and we cannot construct this set using intuitive finite sets without either a dubious "and so on" argument, or a formal axiom which asserts that such an infinite set exists. Freudenthal's arguments are compelling and should be read with care. Numerosity number may be more primitive in some sense than counting number, but the latter is so much more refined and useful that in practice it soon replaces the former. His criticism of Piaget's work on mathematical grounds proves to be sound mathematics.

Numbers are also used for measuring and here Freudenthal introduces the notion of a certain type of abstract system (p. 199) to formalise this concept which is as yet not the object of study in university courses. His ideas are beautiful and logical but they are years ahead of our time for use in schools and stand no chance of acceptance within our culture for much time to come. Nevertheless, if you are a mathematician, read them, they have much to commend them and acceptance by the mathematical community is an essential prerequisite to prevent them from being stillborn.

Meanwhile he has something most important to say concerning the real numbers themselves: should they exist as points on a real line which is then analysed to exhibit the structure of rationals, integers and natural numbers within; or should they be synthesised, being built up from the natural numbers?

A few years ago the gimmick was to synthesise systematically from the natural numbers in advanced courses at college and universities. This synthesis appeared in concrete form (beginning with natural numbers and constructing negatives and rationals) in the Nuffield Primary Mathematics. It didn't work and it needed a mathematician to set the synthesis in its appropriate context. Developmentally the gestalt concept of the real line seems more satisfactory.

It should be pointed out that most of this chapter is written for the mathematician requiring at least the sophistication of a current first year university student although if accepted into the culture, the ideas would permeate down to school level in a suitable form.

In chapter XII we begin to look at the development of the number concept from intuitive methods to algorithmizing and rationalizing. Here the author becomes more interested in the appropriate level of development of the material, considering such topics as use of structured material. His observations are always interesting, often revealing. For instance, he explicitly shows three different methods of viewing the number line:

(1) as points identified with real numbers
(2) as a line labelled by real numbers as coordinates
(3) as a rigid substratum on which numbers act as operators.

In teaching, these are often confused. Freudenthal separates them and gives sound reasons why the second and third should be rejected for didactical reasons. There are many more topics considered, including a request for the return of more practical problems in arithmetic with the suggestion that this might be usefully found in descriptive statistics.

He then continues in like manner, looking at the development of algebra, before attacking the “arrogant shouting of false set theory”. Set theory as introduced in modern mathematics comes in for a considerable beating. Not that the author does not approve of set theoretical concepts; on the contrary, he does. His bone of contention is that the proponents of this basic approach to mathematics have not clearly set their house in order. The concepts of set theory as taught in unnamed school texts are put under close scrutiny by Freudenthal and he proves to be a hard master to satisfy.

His basic thesis is that mathematics starts with number and counting and from these the second in line should be the notion of a function which carries with it the idea of a set. He is most concerned that a function is an active assigning to each element in a set A a unique element in a set B, rather than writing down a set of ordered pairs. He even introduces his own notation for the functional concept translated from logical terminology. Then he considers the case of geometry, mourning the loss of geometry as a study of spatial concepts.

It is here that he discusses Dina van Hiele’s experiments in identifying various levels in the learning process and speaks warmly of early experience with concrete materials for young children. With the coming of coordinate geometry to replace Euclid’s elements in schools, he considers the axiomatic method of the algebraic vector space over a field and the concept of angle introduced in axiomatic terms, rejecting them forcibly as an initial basis of study before the pupil has had a chance to master the basic underlying spatial concepts. In doing so he comes in direct opposition to Dieudonne and the Bourbaki school in France; if I were asked to align myself I would unhesitatingly support the spatial ideas of Freudenthal.

The text continues in like manner looking at analysis. Here he concerns himself with various approaches to the calculus, numerically, graphically, in terms of velocities, densities and so on in an attempt to liven up the ideas on the subject rather than to suggest a specific classroom approach. As a university mathematician he is more concerned with analysing the subject for appropriate presentation at the appropriate level rather than looking at pre-calculus ideas.

He proposes the use of uniform continuity and uniform differentiability as more basic concepts in an analysis course than continuity and differentiability. This is based on the fact that uniform ideas yield the necessary results in a more straightforward manner, which is true jn the case of the particular results cited. But analysis is a notorious subject where if a difficulty is pressed down in one place then another appears somewhere else.

The use of uniform continuity also has its own peculiar lacunae, for instance the product of uniformly continuous functions is not uniformly continuous and the proof that "a uniformly continuous function is bounded on a closed interval and attains it bounds" has a different set of difficulties from the proof for a continuous function. preferred: it is closer to the naive idea that a continuous real function is one whose graph can be drawn freely by hand. Indeed under very mild conditions this conceptually easy notion can be shown to be equivalent to the formal definition; since Freudenthal is less interested in pre-calculus concepts he omits this type of consideration.

The chapter on probability and statistics looks at many examples of the use of these theories with comments on the suitability of the method of approach. Professor Freudenthal has justifiably harsh words to say on the teaching of statistics.

> "I would not recommend the teaching of statistics to college freshmen or high school students ... No part of mathematics is applied with less judgement than is statistics. Statistics as it is usually taught is the worst source of misinterpretation in mathematics. Mathematical statistics though invented to handle numerical data with a critical mind is often used to substitute mechanics for criticism".

Nevertheless we have seen earlier in the book that he sees the use of descriptive statistics as a source of real-life applications of arithmetic. Although he does not mention the phrase "mathematics fraught with relations" in this chapter, it is clear that the coherent framework in mathematics attached to the experience of the student is still a major preoccupation.

His final consideration of the curriculum is the cement which holds it together, namely logic. As we would expect from the rest of the book Freudenthal's view of logic is not just that of a purely formalistic system.

He looks at logic from the vernacular point of view as well as showing how the notion of logical argument has changed in recent history. As usual he subjects the ideas in logic and its notation to his own exacting standards of scrutiny, criticising the current approach and suggesting his own solution. These include a consideration of the central problems including indirect forms of argument, the meaning of implication and the use of quantifiers in a manner which neatly rounds off his study of the curriculum.

CRITICISM OF PIAGET'S THEORIES

In the first appendix Freudenthal adds more criticism of Piaget's work to that which he has already made in the text. He shows clearly that Piaget's misinterpretation of mathematical concepts has led to him misapplying mathematics in some of his models of children's thought. In particular he looks at Piaget's infralogical operations, where the mathematics used is vaguely applied and does not stand close scrutiny. Indeed more could be said than Professor Freudenthal does, for instance, by ambiguous interpretation Piaget claims that his systems exhibit the structure of a group of a lattice by which he means the system is a group where every element is an idempotent. There is only one such structure, the trivial group containing the identity (because if x is an idempotent, then $x^2 = x$, by definition, and on multiplying this equation by x^{-1} we find that x is the identity). His systems contain more than the identity and clearly do not satisfy the required axioms.

Clearly Piaget's explanations are not satisfactory, indeed treated as axiomatic systems they are found to be inconsistent. But perhaps that is the clue. Piaget's experiments show that in the transitional stage before reaching concrete operational thought that the child is capable of coming to inconsistent conclusions. Perhaps the problem is that the prevailing culture did not contain the appropriate model for this and Freudenthal, as part of that culture, dismisses Piaget's misapplication of the wrong model. There is a branch of new mathematics (catastrophe theory) that may contain the seeds of a more appropriate model. This will not render Freudenthal's criticism invalid, but it may put Piaget's work in a more favourable light.

On another attack Freudenthal points out that much of Piaget's research depends on the linguistic ability of the child rather than the understanding of the concepts themselves. He points to deficiencies in the experiments and dismisses the volume of corroborative research in the sentence:

> "even averaging over a large number of observations involving a systematic error does not bring us nearer the truth".It is right that Piaget's work should be doubted and subjected to the closest scrutiny. Even more so the interpretations of Piaget's disciples must be questioned. Misinterpretation of his work has led to many vagaries in primary education. Nevertheless total dismissal of his work does a disservice. It is clear in Freudenthal's book that nowhere does he give a

> consistent view of the development of mathematical notions in young children, beyond a short mention of Dina van Hiele's work and a few other isolated comments. His destruction of Piaget leaves an unfilled gap.

CONCLUSIONS

No one can read this text without it affecting his views on mathematical education. On the whole the mathematical content is aimed at the mathematically sophisticated reader who would need to reformulate it suitably for teaching purposes. Professor Freudenthal's suggestions for improving notation and approach are always thought provoking though some are probably too far removed from the current cultural schema to be absorbed. He has in fact written an advanced text on "Linear Lie Groups" using his own notation which proved unintelligible to many experts.

It is strange that a man steeped in knowledge of history should try to replace the rich but inexact language of developing mathematics by a new form of esperanto familiar to none. Indeed Professor Freudenthal is so concerned with precision in thought and language that he never advocates that the reader should learn to live with the vagaries of mathematical notation which are inextricably wound up with its development. Inexactitude of language is even a necessary part of the mathematical way of thought. As we become familiar with an object in context we learn to refer to it in less precise terminology because the context keeps the record straight. (The types of notation which can be used as one develops a concept are beautifully described in Professor R. Skemp's book *The Psychology of Learning Mathematics*.)

Professor Freudenthal's approach to the subject is that of an excellent mathematician in the context of an ad hoc approach to psychology involving "honest examination" of the way one learns concepts. If there is a weakness then it is one of structure of which the author is himself aware because he mentions it at the outset. No chapter begins with an outline of material to be covered, no chapter ends with a summary of contents and any chapter is liable to contain asides relevant elsewhere.

This is where an index would have been so valuable. For instance, the criticism of Piaget in the appendix does not include material earlier in the text, but there is no way of discovering where this material is to be found without skimming through the whole book. It appears in the most unlikely places.

Having said this I feel that my library is immeasurably richer for having this book which covers a path of experience through mathematical education, steering clear of the extremes of devotion to developmental psychology on the one hand and excessive abstract formulation on the other. At last a real mathematician has attempted to give a coherent view of his educational philosophy in a comprehensive and thoughtful manner.