Cognitive Conflict and the Learning of Mathematics

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ABSTRACT: Understanding in mathematics often occurs in significant jumps, accompanied by a clear sense of comprehension, ratter than a smooth, steady process. Lack of understanding, on the other hand, may leave the individual in a general state of confusion, unable to pinpoint the difficulty. If we consider these phenomena to be a result of brain activity of the nature of a dynamical flow on a manifold, then this suggests a model which encompasses these various aspects of understanding mathematics.

As the learner restructures his mathematical schema to understand these ideas, cognitive conflict is bound to occur. It can give rise to path-dependent logic, in which the learner can give different answers to the same questions depending on the path of approach to that question. At this stage the learner may restructure his ideas and rationalise them in a manner appropriate for a short term gain but inappropriate for long term schematic development.

In this paper we give case studies of conflict and path-dependent logic, and explore possible links with catastrophe theory which afford hopes for a global model of the learning mechanisms of the brain.

Introduction: Mathematical Models

One of the most important roles of a mathematician is to describe natural phenomena in terms of a mathematical model and to apply that model to make predictions of the behaviour of the natural phenomena under given conditions.

This occurs in classical applied mathematics, in physics and engineering, and also in the wider realms of applicable mathematics in statistics, computing, economics, biology and so on. A significant area in which mathematicians experience difficulty is in the actual teaching, learning and understanding of the subject. If the mathematical physician is to heal himself, then perhaps he should look to the model theoretic aspect to show the way. This is not to say that mathematical models have not been used before in the psychology of learning mathematics, of course they have. Statistics have often been used to underpin educational research though one wonders if this has always been used to the best effect, or whether it is best used at all. It is one thing to show a statistical link between smoking and cancer but in the end the best possible verification is a biological one to demonstrate the actual effect of smoke

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on the lungs. The latter has yet to be satisfactorily demonstrated; in a way there is a similar problem with the psychology of learning mathematics. The actual mechanism of learning mathematics is going on in the brain and we still know very little of the way the brain functions.

Other mathematical models have been used to describe the developmental processes. For example Piaget made great use of group theoretic ideas, with his notion of a grouping, and of the INRC group. The first of these is mathematically deficient (though attempts at patching it up have been made by Wittman), and the second notes two phenomena which can be described by the Klein four group and postulates linkages between the two. Piaget's brilliant research work also shows up the problems that can occur in the interpretation of his models by others. It is one thing for Piaget to note stages in the development of a concept, it is another for those stages to be used in a trivialised way for teaching purposes. An amusing misinterpretation of Piaget's theories might lead to the deduction that a child cannot be taught conservation until he has acquired it himself, and when he has acquired it, it need not be taught, so the teacher's role is peripheral beyond providing a suitable environment for learning. The trouble is that many educationalists have actually made such a deduction and the child-centred nature of certain schools of thought in the sixties and seventies has undervalued the role of teaching. At the same time inconclusive, and sometimes contradictory, statistical evidence about the role of the teacher has not given any clear and incontestable lead as to how mathematics is learnt and how it might be taught.

The problem with a model is that, by the very nature of things, it concentrates on certain aspects of the situation and neglects others. By this process of simplification it can possibly predict quite accurately those aspects on which it concentrates, whilst being partially, or totally inaccurate in other respects. The existence of a model is not sufficient in itself, it also requires an understanding of its limitations and the manner in which it is to be interpreted.

If we are to get to the very core of the problem of learning in general, and learning mathematics in particular, then we must eventually understand the mechanism of the brain itself. Until such an insight is possible we must be content with describing the externals of the phenomenon, but at least we can do this in a way which gropes towards an ultimate description.

There are many ways of modelling, or simulating, some aspect of brain activity. One model which might be particularly appealing to mathematicians is that described by Zeeman [10]. In Zeeman's own words, it is a "medium-scale model, midway between small scale neurology and large scale psychology." He explains this by comparing the levels of modelling the action of a knee joint; small scale would involve description at a cellular level, medium scale would simply picture the lever action and muscular action, whilst large scale would place the knee in a wider context, seeing its relationship with other parts of the body, its use in walking, in sport and so on. Zeeman regards the brain as a dynamical system, say by measuring the potential across each cell and recording the values as coordinates of a point in a space of very large dimension. The position of this point would determine the thoughts and actions of the person and there would be a relationship between the model and the behaviour of the person, which would not preserve measurement in any sense, but would reflect discontinuities in measurement. This would suggest that at least some of the changes in brain activity might be modelled in terms of catastrophe theory. For further detail, see Zeeman's exploratory paper [10], which makes a number of interesting suggestions, and for other aspects of catastrophe theory see [11].

It is not pertinent to make excessive claims for the present state of play in using catastrophe theory in the psychology of learning mathematics. This is not to say that we cannot look for certain types of behaviour which we might expect the theory to suggest; it does mean that we must take care not to overstress the role of catastrophe theory to "explain" and to "predict" behaviour in our present state of limited understanding.

Background to investigations

Let us turn to the investigations which led to a catastrophe type interpretation. Originally the intention was not with this brief in mind at all. It began with the simple leading question

> "Why can't students do analysis?"

coupled with the aim to develop a long-term learning schema for calculus/analysis.

Various investigations were initiated to try to get to grips with the problem, to formulate it more specifically and to set up objective tests in the hope of getting answers. Was it to do with the short-term memory capacity to handle the long definitions, or with ability to manipulate remembered facts mentally or whether the students were geometrically or analytically biased with their thinking, and so on? In these investigations the most interesting (in a purely personal interpretation) were none of those mentioned. The intriguing topics concerned the students own interpretation of key concepts, their use of words, the way they built up concepts and the conflicts which occurred as they restructured their schema. It is perhaps a personal preference that the interesting points did not (at first) involve statistical comparisons of large groups, rather the long-term development of individuals. To coin a colloquialism, “that is where the action is”, in the mind of each individual student.

The investigations were developed in conjunction with Professor R. L. E. Schwarzenberger, who allowed questionnaires to go out during his first year analysis lectures. Each student was given ten sheets, numbered one to ten and each student marked their own sheets with a personal number. This gave freedom for the lecturer either to give pre-arranged tests, or, on the spur of the moment to ask a question (or questions) of every student in the audience. In addition a number of volunteers answered a questionnaire each week.

Faux amis

Certain interesting phenomena occurred. In the first place we found an amazing variety of interpretations of well-known mathematical words, especially in terms of intuitive ideas before the words were “formally” defined in the lecture course. Some of these are explained in [8], [9]. They included words like ‘complex number’, ‘real number’ ‘limit’ ‘continuous’, ‘infinity’, ‘proof’, ‘some’, etc. In many cases students gave conflicting explanations of words. For example a student might say √2 was a real number, but not a complex number, then later, when asked what a real number is, reply “a complex number with imaginary part zero.”[1]

The general purpose of the questionnaires was to ask students questions in areas of conceptual difficulty where we would not expect understanding. We explained that the questions were intended to cause confusion and asked for the most honest answers possible without allowing the exercise to cause distress. One student withdrew, writing to say that the questions forced her to commit answers to paper which didn’t really convey what she actually thought, so it was pointless to carry on; a timely warning that what a student says need not represent the actual process of thinking. The use of the word ‘concept’ was also exposed to suspicion. Mental thoughts are often so nebulous or tenuous that to crystallise them with a word like ‘concept’ gives the thought more precision or permanence than it actually has. A given word could take on different shades of meaning according to the context. Skemp [4] has already remarked on such a phenomenon occurring in different schematic contexts and coined the term ‘faux amis’. One of his examples was that of a ‘field’ which is interpreted in different ways in starkly contrasting schemas. The same phenomenon may occur in what might ostensibly be

[1]Mathematicians also unthinkingly exhibit this conflict. In set theory the real number x is distinguished from the ordered pair $(x, 0)$, but in complex number theory the real number x is identified with $x+i0$ which is the ordered pair $(x, 0)$. Perhaps √2 is real, but √2+i0 is complex.

described as a single schema especially during the restructuring process or when conflict is present and the schema is unstable. In fact, although the words ‘concept’ and 'schema' continue to be useful (at least for the moment), the precise interpretation of these words is also a matter of conjecture.

Many examples of differing interpretations of words have been given (e.g. Skemp, [4]), so these need not detain us long here. The interesting point in developing long term learning schema is to see if the words evoke certain mental responses which then interact in possibly unforeseen ways in the learning process. A simple example is the use of the word ‘some’. On a questionnaire students who had just arrived at university were given a list of statements and asked to say whether they were true or false. The statement:

“some rational numbers are real”

was adjudged by a majority of students to be *false*. (Clearly *all* rationals are real). A follow-up questionnaire tried the following:

> Let *S* be the set of numbers 19, 3677, 601, 2, 257, 11119, 7559, 12653, 11177. Without doing any calculations, say whether the following statements are true or false:
>
> (l) some numbers in the set *S* are prime,
>
> (2) some numbers in the set *S* are even.

The first of these was universally adjudged true (some are visibly prime but the rest are, to the student, unknown). In fact *all* the numbers are prime. The second was considered false by most (probably) because of the plurality of the verb), but certain students were happy to consider it true. A colloquial interpretation of the word ‘some’ in many cases seems to be that it means “two or more of a given set, but without the certain knowledge that it is all of the set.” Shades of meaning vary from person to person.

Other examples of this nature are given in [8] and [9]. They were used in preparing a text [5] which aims at helping students arriving at university to restructure their basic ideas in the subject, whilst introducing the fundamental mathematical notions.

Point nine recurring

One of the more fruitful questions asked students was:

> “Is 0.999.... (point nine recurring) equal to one, or just less than one?
> Give reasons for your answer.”

A majority thought that it was just less than one, but whichever answer was given, the student. often supported it with infinitesimal reasoning. The details are given in [9]. A set of questions was devised to approach the question another way. Decimals were written down in class and the students were asked to convert them to fractions in simplest form, for example 0.5 = $\frac{1}{2}$. The fractions given were 0.25, 0.05, 0.3, $0\cdot\dot{3} = 0\cdot 333\ldots$, $0\cdot\dot{9} = 0\cdot 999\ldots$. The last two were called 'nought point three recurring' and 'nought point nine recurring' as they were written down. A number of students who had claimed that $0\cdot\dot{9}$ was less than one now wrote $0\cdot\dot{3}$ = 1/3, $0\cdot\dot{9}$ = 1. Some saw the conflict with $0\cdot\dot{9}$ = 1 and either crossed it out, or it was just less, and returned to $0\cdot\dot{3}$ = 1/3, altering this to say 'just less' as well.

The answer to the original question depends therefore on the path of approach. One might argue that the question depends on the context and that in different contexts the same words have different meanings. This is precisely the point referred to above and has given rise to the hermeneutic approach to the psychology of learning mathematics [12], [2]. This does not conflict with a catastrophe type interpretation. An analogy may be given by considering the canonical projection $p{:}C \to X$ where X is a topological space and C is its covering space; different paths in X may end up at the same point, but lifting them gives different endpoints in C. (An example would be the circle $X = \{e^{i\theta} \in \boldsymbol{C} \mid \theta \in \boldsymbol{R}\}$ and the covering space C = $\boldsymbol{R}$, with projection $p : C \to X$, $p(\theta) = e^{i\theta}$. The paths $\gamma_1{:}[0, \pi] \to X$, $\gamma_2{:}[0, \pi] \to X$ given by $\gamma_1(t)=e^{it}$, $\gamma_2(t)=e^{-it}$, have the same endpoints but the maps $\lambda_r : [0, \pi] \to C$ which lift γ_r (r = 1,2) (i.e. $p\lambda_r = \gamma_r$) are $\lambda_1(t) = t$, $\lambda_2(t) = -t$, which end up at the different points $\pi, -\pi \in \boldsymbol{C}$.) Some of these different interpretations may be distinguished by different paths of approach, using probing questions to determine shades of meaning.

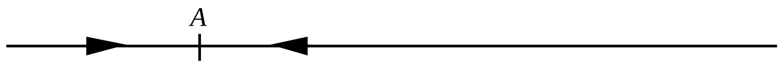
Catastrophe theory

The phenomenon that different paths of approach may give rise to different output is a phenomenon of catastrophe theory, as it is in other theories. One of the distinguishing factors in catastrophe theory is the existence of discontinuities, or sudden jumps in behaviour when certain paths are taken. In catastrophe theory a potential function is being minimised under given constraints. For certain constraints there may be several possible positions of local minimum energy (stable) and local maxima (unstable). As the constraints vary smoothly, so do the maxima and minima. But it is possible under changing constraints that a minimum which was tenable coalesces with a maximum and the minimum is no longer tenable, so a jump is made to another position. Full mathematical details will be found in the literature (for example in [11]).

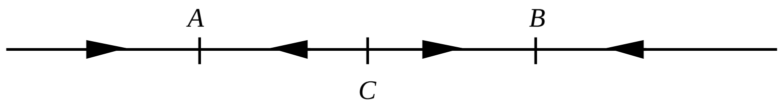
Applications in the psychology of learning mathematics are still in a very rudimentary stage. A preliminary discussion occurs in [8]. A full catastrophe interpretation of the state of the brain as posed by Zeeman would require a large dimension manifold (of the order of 10^9

dimensions). Nevertheless a one-dimensional model, drastically oversimplified, can help represent the restructuring of schema and illustrates clearly the basic catastrophe idea.

Suppose an attractor *A* is on the real line with flow lines into *A*.

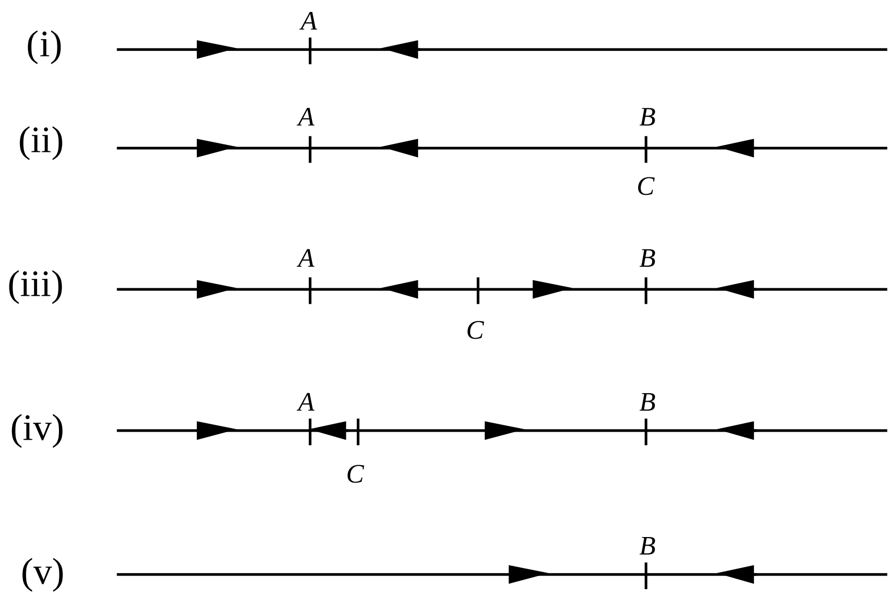


With two attractors *A*, *B* and a flow on the line, we have a repeller *C* in between.



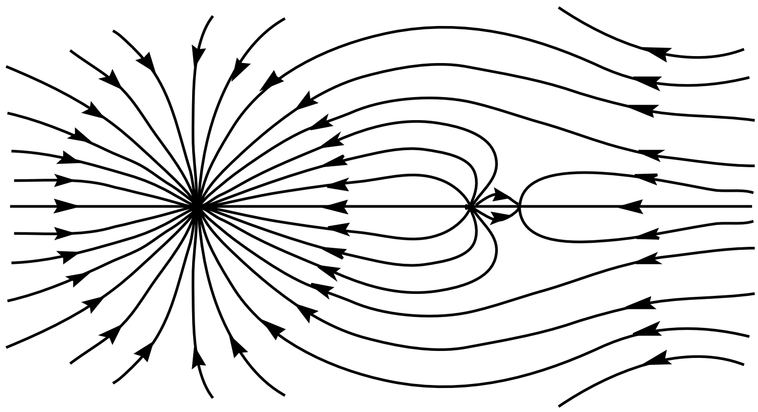
A variable point on the line would move in the direction of the arrow and end up at either *A* or *B*. (A point at *C* would stay there, but a small perturbation would cause it to flow to *A* or *B*, hence *C* is *unstable* whilst *A*, *B* are *stable*.)

A trivial, but useful model of the restructuring of schema, starting with an old concept (or schema) *A*, the formation of a possible alternative *B* with conflict *C* and the eventual resolution with the annihilation of *A* can be represented in the sequence of diagrams (i) – (iv) considered as a smooth transition.



At stage (i) the old concept *A* is firm, but changing constraints cause the development of a possible alternative *B* and the conflict *C*. Initially (when *C* is close to *B*) the stronger *A* holds sway and only when a point is very close to *B* is this sufficiently strong to draw it in. As *B* grows in strength, *C* moves nearer *A*, and there is a genuine transition stage (iii), where either *A* or *B* is tenable. Depending on how close a point is to *A* or *B*, it will be drawn into the one whose influence is stronger. The restructuring of ideas continues until the conflict is resolved by cancelling it out with the annihilation of *A*, leaving *B*.

This is a trivial model, even a poor one in a number of respects. Like any model it has possibilities of misleading interpretations. For instance at stage (ii), it- might seem possible that any point to the right of *B* would be drawn into *B*, giving it a strong sphere of influence. When *C* coincides with *B*, or is close to it, a small perturbation will dislodge a point from the minimum *B* and, once beyond *C*, it will flow into *A*. Thus in practice, under a perturbation *A* would remain the dominant attractor. *A* two dimensional model would show this better:



In [8] the one dimensional model is shown as a cross section of the cusp catastrophe, bringing it in the sphere of applied catastrophe theory. It is still in the position where we must be very wary to use the model to ‘explain’ or ‘predict’. However, certain facets seem familiar. For instance the sudden leap of understanding, accompanied by a clear sense of comprehension as one understands a concept. On the other hand there is the general feeling of confusion, unable to pinpoint the difficulty as one tries to comprehend a new idea in the presence of conflicts in the schema. Indeed the conflicts may be so strong that the dynamical system of the brain is drawn along unsuitable paths leading to rejection and inability to understand.

Other examples

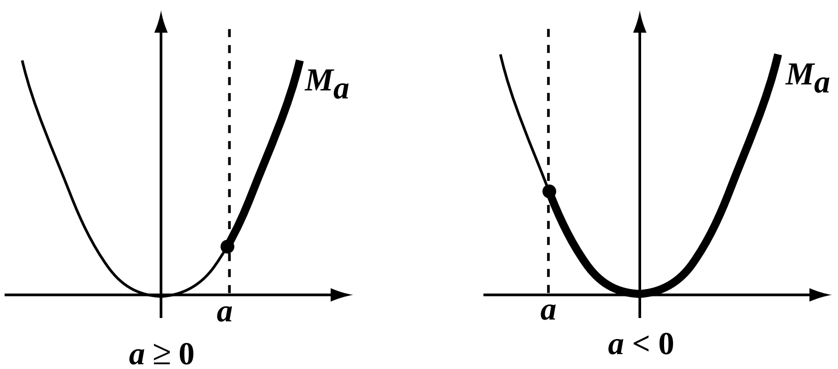
The work of Piaget abounds in examples of transition. The case of the transition to the concrete operation stage is discussed in [8] and in [6] a corresponding transition is predicted within the formal operational period. An example not quoted in these papers is the Piagetian observation of the child trying to replicate a set of items in a line, but in reverse order. He begins satisfactorily, starting at the other end of a duplicate line, but on reaching half way, loses track and for the second part of his line he replicates the beginning of the first line in the same order.

Often a child or student. learns a technique in a restricted situation and is confused in an extended situation where the restricted technique does not work. For example, a lack of understanding of decimals might lead to the restricted notion that the number after the decimal point represents tenths. This is fine with 0.l, 0.2 and so on, to 0.9, but what about 0.10 and 0.11? Is 0.11 equal to eleven tenths? I am grateful to Margaret Brown for this example. It illustrates how limited preconceptions can prevent full understanding if the conflicts are not resolved.

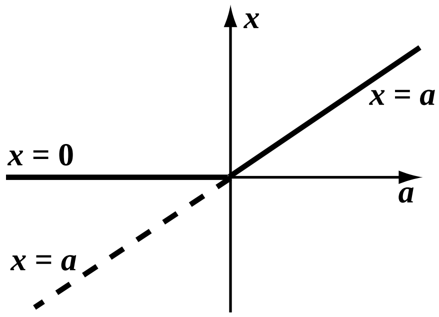
The use of catastrophe theory

In our present state of limited understanding, there are several likely pitfalls to be avoided. Until we have a clearer understanding of the brain dynamic, he must be careful not to overstress the roles of "explanation" and "prediction". The model would suggest a complex structure involving complicated discontinuities. Some of these may be described in a simple (or simplistic) way involving an elementary catastrophe, for example the cusp catastrophe as in [8]. In a dynamic situation, this could change to another catastrophe. For example the cusp (which involves a jump between two possibilities, with no compromise) may develop into a butterfly (which has a compromise possible).

Over-trivialisation could lead to an incorrect interpretation. We must note that the relationship between the brain and a postulated model may be far from perfect. The state of the brain need not even relate in a one-to-one way with the sensory inputs and outputs. There may be filtering processes going on which lose information en route. In this way a variant of catastrophe theory may be appropriate, say one which corresponds to a catastrophe theory on a manifold with boundary (the filtering corresponding to the cut-off at the boundary). As a brief example of the latter in process (which arose in discussion with Ian Stewart) consider the manifold M_a with boundary $\{(x, y) \in \boldsymbol{R}^2 \mid y = x^2 \text{ and } x \geq a\}$ and find the maxima and minima of the height function $h : M_a \to \boldsymbol{R}$ where $h((x, y)) = y$. For $a \geq 0$ there is only a minimum on M_a, given by $x = a$, but for $a < 0$, there is a minimum $x = 0$ and a maximum $x = a$.



Regarding a as the control variable, varying a and seeking the values of x for which h is a minimum (stable), or maximum (unstable), we find that plotting the possible values of x against a we get the following diagram, in which the unbroken graph would be a likely (stable) position and the broken graph unlikely (unstable):



Thus we may have a model in which to a certain stage there are two possible outcomes ($a < 0$) one likely, the other unlikely, but after a certain point ($a = 0$) the previous likely one vanishes and the unlikely one becomes the only possible behaviour. Klahr and Wallace have observed such a phenomenon (though without any reference to catastrophe theory) in the counting of a collection of objects. For less than five or six objects there are two ways of finding the number, one by just looking, (likely) and the other counting (unlikely). Beyond five or six the method of just looking breaks down and only the counting method is available.

Long-term learning schemas

In [6], a beginning was made in the attempt to develop a long term schema for calculus and analysis. The plan was to analyse the schematic ideas that students might develop at various stages and work out possible methods of approach to the subject which would help student understanding. The work, which is still in progress, began with an initial pilot course at Warwick University in the Autumn of 1975, followed by further investigations in Autumn 1976.

The work will be continued in 1977. Lecture notes, [7], were written for 1975 which attempted to take various concepts, postulate an intuitive way of looking at them then restructure the schema to give a formal approach. For example the notion of drawing a graph without taking the pencil from the paper was shown to be equivalent to the formal ε–δ definition and other notions such as limit of a sequence, limit of a function, differentiation etc. were treated in an overall schematic way which developed from intuitive ideas to rigour.

With many students the approach was favourable but a number were confused by the “explanations”. The obvious fact, which is a drawback in writing any book in a developmental way, is evident. To be able to communicate with the reader one postulates his level of understanding and then restructures this to develop what is required. The major problem is that to develop a formal theory T, one may postulate that the student is at stage A, then explain the transition from A to T. If the student is at stage A, that is fine, but if he has some other totally different viewpoint B then the text becomes difficult. He must first of all make the transition (if he can) from B to A before he can understand the move from A to T explained in the book. In a lecture, the tone of voice of the lecture, the manner in which he explains things, the method he builds up the theory, all these can help the student find his bearings. But in a book the cold print does not always give such clues.

Different students have different problems, so if student S has problem P in understanding because of his schematic development, but student Q has no such problem, it does not help Q to be given a long drawn out discussion of problem P. It could make matters worse. A text book which explains all the problems different students might encounter would be unreadable, so the notion of a single text that is based on a long-term learning schema is a very difficult one to conceive. A working approximation is all that one can hope for, but it can be helped considerably by investigations into the possible states of students' schemas before a concept is taught. When we speak of a long-term learning schema, we are in fact using the word schema in a strategic sense rather than a psychological sense. It gives a basic plan of attack, cognisant of the problem which students are likely to encounter. Rather than speaking of "long-term learning schemas" as in [6], it would be better to use a description such as "long-term learning framework", meaning a long term plan of attack broadly suitable for most students. A suitable method of approach has been intuitively employed in many institutions for years, a broadly conflict-free explanation of the general theory in class or in a lecture or from a book, followed by individual help to sort out individual difficulties in tutorials.

The role of the teacher

In the learning process, the role of the teacher is of paramount importance. Programmed learning, work cards and so on may be an effective teaching substitute in certain circumstances, but the essential role of the teacher is helping in the schematic restructuring of the student. The occurrence of conflict in the mind of the learner will be apparent immediately to the sensitive teacher. The simplest manifestations are confusion, annoyance, fear, or just a dull lost look in the eyes. It would be wrong to separate these emotional reactions from the cognitive side of learning. They are all signs of the state of the brain. In terms of a catastrophe interpretation they may help us to realise the nature of the mental blockage, an unsuitable line of thought, a catastrophic leap, or even path-dependent decision making. Then it is the job of the teacher to resolve the conflict in a suitable manner. Continual explanations along the same track to emphasise the required idea may not help, because the students train of thought may be leading elsewhere. The learner may even not know where the problem is, but the experienced teacher may be in a position to help by selecting the right approach. These ideas of the excellent teacher are as old as history itself. But then the history of science shows a continual reinterpretation and enrichment of earlier ideas. Perhaps a catastrophe interpretation will lead to placing the well-tried skills of the teacher within a theory which describes the mechanism of the brain itself. In the meantime, by being sensitive to the possible conflicts in the mind of the learner learning new mathematical ideas, we may find a practical way of understanding mental blockages in the learning process.

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