Conflicts and Catastrophes in the Learning of Mathematics

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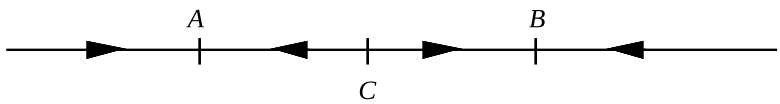
In this journal a year ago [6] an analogy was suggested between Piaget's transitional phase into concrete operational thinking and a later stage in formal operations when students experience difficulties with mathematical proofs. It is the purpose of this article to explain the catastrophe theory model in greater detail and also to report the results of a test performed on first-year university students at Warwick University in collaboration with Rolph Schwarzenberger. I wish also to record thanks to lan Stewart, Robert Zimmer and Elizabeth Hitchfield for helpful discussions and incisive comments. The lesson of catastrophe theory has wide ramifications which enhance the notion of schematic learning theory [5] whilst warning against too rigid an interpretation of a schema as some sort of ordered graph of concepts. Rather it suggests that a schematic structure is a dynamical flow involving attractors and repellers. Roughly speaking we may classify much of the "new mathematics" as a genuine attempt to explain the concepts in a thoughtful and consistent manner with understanding. This might be considered the positive side of the picture. Despite the care and enthusiasm involved, the approach has not been a hundred per cent success. The catastrophe theory approach suggests strongly that we should look at the negative side as well. This involves considering not only the attractors formed in the brain's dynamical system, building up new concepts in a coherent schema, but the repellers which prevent suitable linkages from being made. At a time of national (and international) concern over the inability of adults to understand seemingly simple mathematical concepts, we should be devoting a large part of our energies to searching for a more balanced and complete learning theory.

A simple conflict model

Let us begin with the simplest of mathematical models, a one-dimensional system with two attractors on a line and the resultant flow.

Between the two attractors *A* and *B* we have a repeller *C* from which the flow emanates.



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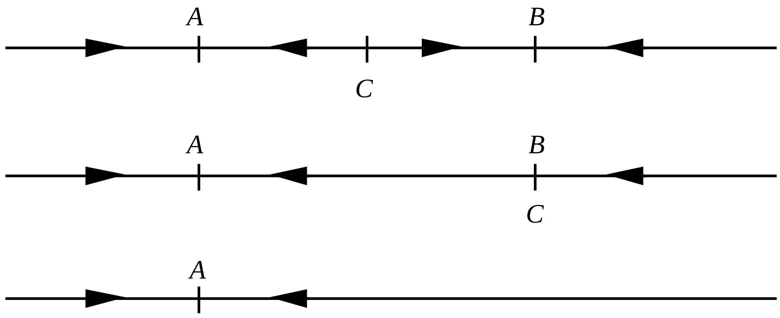
If we consider the gravitational action of the earth and the moon as attractors, then the repeller between them is just a mathematical fiction marking the balancing point of zero gravity. In other instances we may conceive *C* as being a genuine repellent source. Mathematically there is no distinction between these since we are concerned with the flow and what happens to a movable point *P* on the line under the influence of the flow. Near *A* the point *P* will be drawn into *A*, near *B* it will be drawn into *B*. If *P* were placed at *C* it might stay there, uncomfortably balanced under opposing forces. All three points *A*, *B*, *C*, are equilibrium positions, with the difference that whilst *A* and *B* are stable, *C* is unstable. A small displacement of *P* from *A* or *B* will see a return to that position under the imposed forces but a small displacement from *C* will lead to a movement either to *A* or *B*.

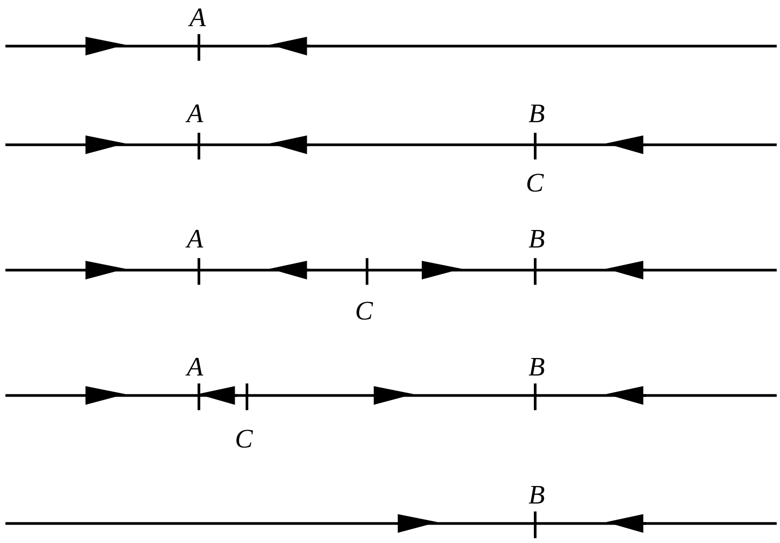
Now let us vary the picture with time and let C move towards B in such a way that the two coincide and annihilate each other, leaving just a single attractor A.

(i) (ii) (iii)



We can consider this variation in reverse via sequence (iii), (ii), (i), where we see initially an attractor *A* alone, then a coincident attractor *B* and repeller *C* created elsewhere, which separate, giving the system consisting of two attractors *A*, *B* with a repeller *C* in between. Let us take this reverse process and follow it by the coincidence of attractor *A* with the repeller *C* as in the following sequence:

(1) (2) (3) (4) (5)



Think of this transition occurring smoothly, starting with a single attractor *A*, the creation of another attractor *B* and repeller *C*, with the eventual annihilation of *A* by *C* leaving the new attractor *B*. Now what happens to a variable point *P* under the action of the dynamical flow? In stage 1 it blissfully homes into attractor *A* and stays there. Any slight disturbances and it returns happily to home. At stage 2 a blot appears on the horizon, as yet too small to cause any problems, but by stage 3 the conflicting tensions are more evident. A small displacement of *P* from *A* and it is drawn back again, but a large displacement in the appropriate direction leads to a jump to *B*. Even if the variable point stays at *A*, the inexorable march of time towards stage 4 makes *A* less and less tenable. As *C* moves ever closer to *A*, a small and smaller displacement is necessary to cause the jump to *B* until, beyond stage 4, *A* becomes no longer an equilibrium position, only the attractor *B* remains to draw in the variable point.

An interpretation in cognitive development

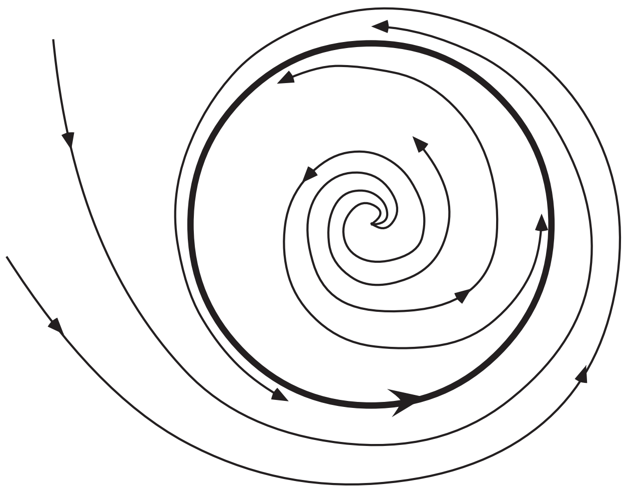
This simple dynamic model may be considered as a qualitative description of mental activity which can happen in forming new concepts.

The learner at stage 1 of development has a concept *A* available. As he develops, new situations occur for which *A* becomes less tenable; an alternative concept *B* appears and with it is born the conflict *C* between the two. He is now in the transition phase between stages 2 and 4. In situations close to *A* he homes in on it but a change in context displacing him nearer *B* and he may be drawn to that. In this transition phase he is therefore likely to come to differing conclusions depending on the approach and context which force him towards *A* or *B*. This may be accompanied by a state of confusion which in extremes may manifest itself as anger on the one hand or docile submission on the other. By being subjected to suitable disturbances he may suddenly jump in his decision making processes from *A* to *B* (or vice versa).

The sudden jump from *A* to the (more suitable) concept *B* (when "the penny drops") may be accompanied by a sense of pleasure and achievement. Nevertheless, until the contradiction *C* is removed together with *A*, regression to the previous concept *A* remains a possibility. In the model the elimination occurs at stage 4 and beyond. Here the contradiction *C* is brought squarely up to concept *A*. The learner resolves the conflict by seeing the inappropriateness of concept *A* in the enlarged context and *A* and *C* are eliminated, leaving the new concept *B*. The conflict *C* which initially acted at stage 2 and just afterwards to prevent access to *B* now moves over to annihilate the old concept *A*.

Incidentally, in this description the word "concept" may be replaced throughout by "schema". It was Dr. R. Zimmer who first pointed out to me that a schema may be considered globally as a concept and a concept may be dissected to reveal a schema. In this context we see a schema *A* at first an adequate course of action, then in an enlarged context it proves no longer appropriate. The development of an alternative schema *B* and the conflict between them is only resolved when the learner finally comes to terms with the situation and resolves the conflict by realising where *A* is inadequate eliminating the conflict and the inappropriate use of A.

At this stage our simple one-dimensional model is becoming simplistic. For instance, we may require a flow on a subset of a higher dimensional space involving many attractors and repellers. This dynamical flow will vary with time, attractors and repellers being both annihilated and generated. Furthermore the attractors and repellers that occur need not be points but may themselves be stable flow cycles, as in the solution to the Van der Pol differential equation [3] where there is a stable circular cycle and stream lines spiral into it from inside and outside.



Towards a general theory

The search for such a model has been suggested by E. C. Zeeman [7] as a "medium scale" model of brain activity midway between small-scale neurology and large scale psychology. It would involve catastrophe theory and its most vital factor is that it suggests that at least some of the changes in brain activity can be modelled by elementary catastrophes. One of these, the cusp catastrophe, has already proved useful in applications and we shall see shortly that the model we have just explained can be visualised as a path through the control space of the cusp catastrophe.

Before considering any of these ramifications, we begin to get a picture of how a changing dynamical flow might represent the accommodation of schema. If such a flow represented the current state of mental schema and a new concept were introduced, the existence of repellers may bar the way to latching it on to an appropriate attractor and deflect it to an unsuitable one. The resultant dynamical flow may generate other attractors and repellers in a way highly unsuitable for new tasks.

A number of standard phrases graphically illustrate the general ideas: “the penny drops” as the mind jumps to an attractor, “there’s a mental block” as the existence of a repeller prevents access to a desired attractor. Indeed these would also tie up with why a student in difficulty cannot put his finger on the precise nature of his problem. The student may be at attractor A, the teacher attempts an explanation of attractor B but the student is unable to pass beyond repeller C which lies in between.

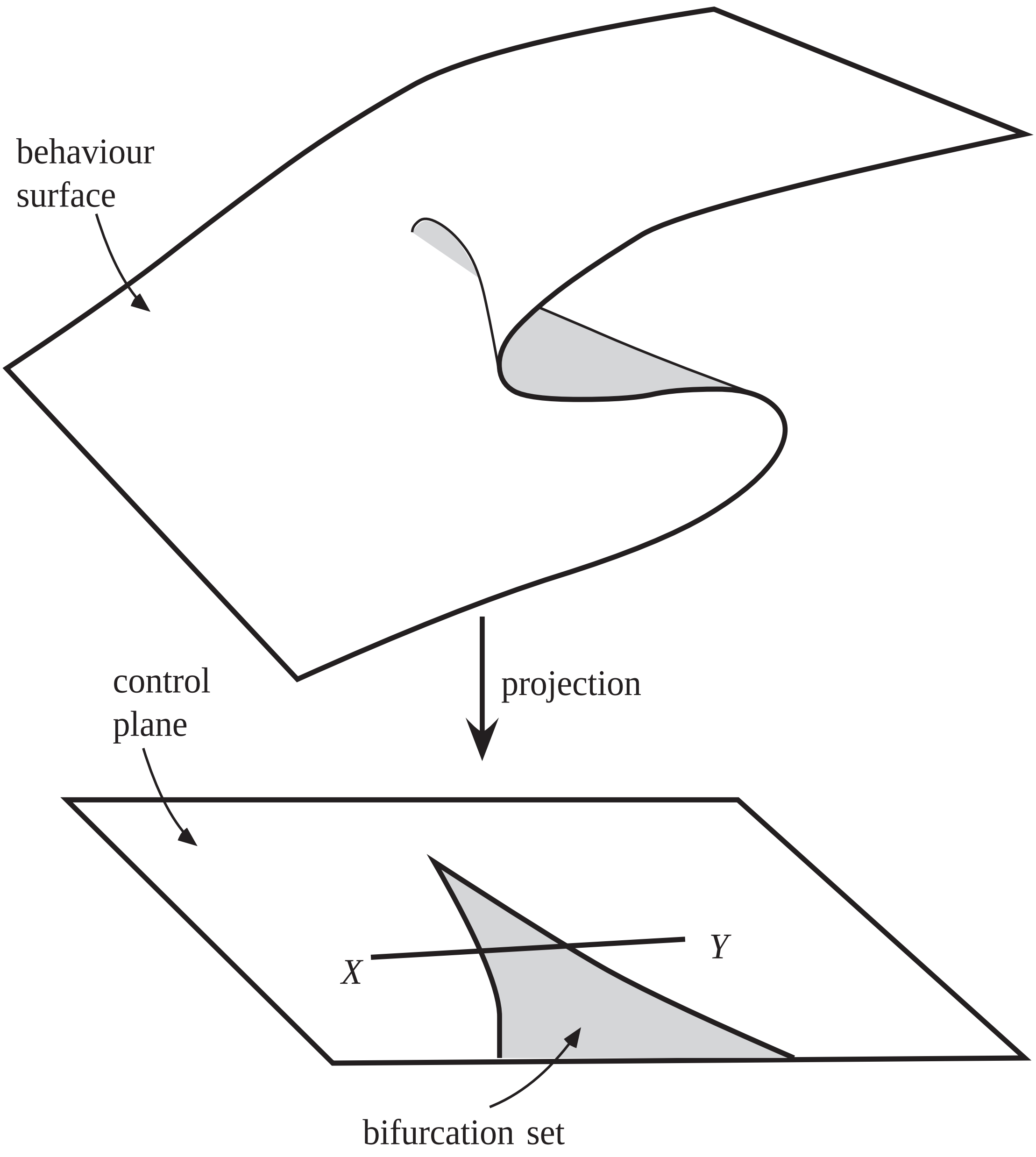
A local explanation of B will not help the student, he cannot get there, the conflict C must be resolved before a lasting understanding is possible. Meanwhile he cannot say what his problem is. The analogy is with a fixed magnetic pole as one approaches it with a moving pole. If the moving pole is unlike the fixed one, there is an attractive situation and we know where it will end up. However, if the moving pole is like, there will be repulsion and the moving pole will be pushed somewhere away from the fixed one. In an analogous manner we might expect an attractor to produce a precise feeling of understanding when it is reached, but a repeller to cause a vague feeling of unease.

Flow Schemas

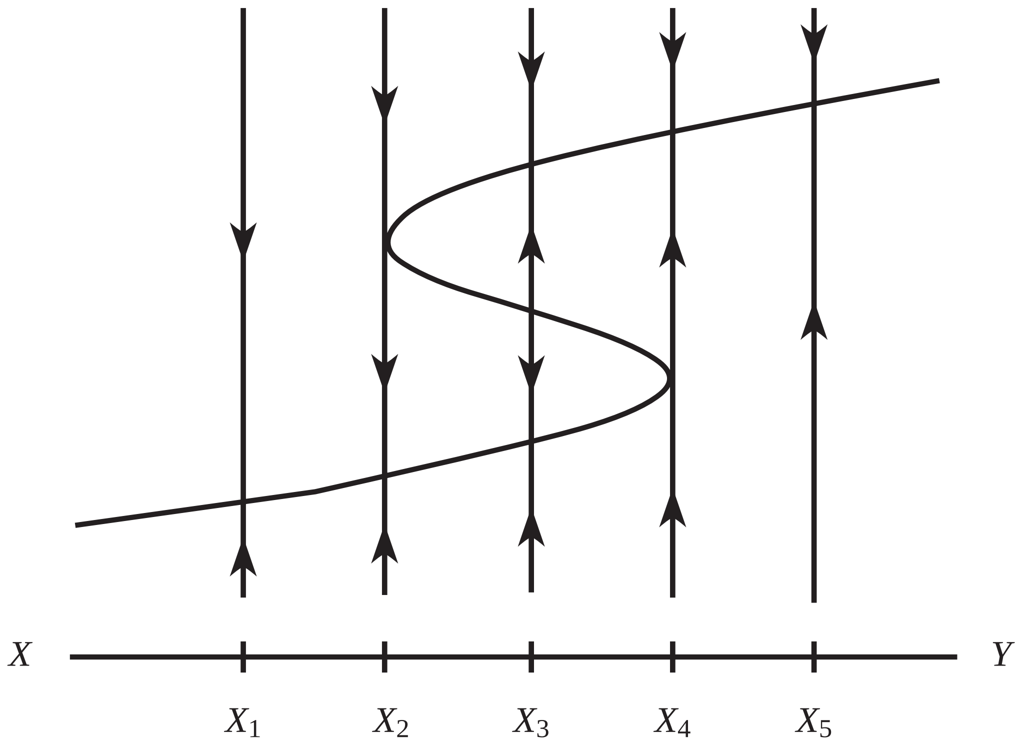
Suppose that the total activity of the brain is represented by a changing dynamical system in a large dimensional space M. Portions of this activity will represent coherent schemas, in fact we will define a flow *schema* to be a subset S of M closed under the dynamical flow. This simply means that if $x \in S$, then under the flow x remains in S. In the Van der Pol picture there are three such subsets which can be clearly seen, the stable circular cycle C, the disc D including its boundary C, and E, which is the exterior of D together with C. Using Bob Zimmer’s idea that “a schema is a concept is a schema” we can now concentrate on particular aspects of the picture. The idea is crystallised by Richard Skemp’s excellent word “varifocal”. If we consider any flow schema F, then it may have other flow schemas within it: F_1, F_2, … . We simply form the quotient space formed by collapsing any of F_1, F_2, … to a point. For example, in the Van der Pol flow we have $C \subset D$ and collapsing D to a point, C/D is topologically a sphere, similarly E/D (collapse D to a point in E) is topologically a cone and $(C \cup D)/D$ is a cone stuck to a sphere by its vertex. The quotient space has a flow induced on it and we will refer to this system as a flow schema also. In performing such identifications it is essential that the spaces collapsed are closed under the flow otherwise there will not be a coherent flow induced on the quotient space. A change in dynamical flow on F may therefore render a subset F_1 initially closed under the flow no longer stable. The quotient space F/F_1 no longer has a coherent flow and we must return to the larger flow schema F to consider the revised situation, possibly finding new closed subsets G_1, G_2, … allowing us to form new quotient spaces. This is the essence of the varifocal theory with the added ingredient of the flow patterns.

The cusp catastrophe

Having entered the realms of speculative visions, it is as well to return to earth with a specific example of catastrophe theory in action. The cusp catastrophe is well described in the literature. A popular account occurs in Christopher Zeeman's article in the Scientific American [8]. For our purposes we can picture it in terms of a folded surface in three-dimensional space called the *behaviour surface* and a projection down on to a plane called the *control plane,* as in the diagram. The *bifurcation set* is the set of points in the control plane above which there is more than one point on the behaviour surface.



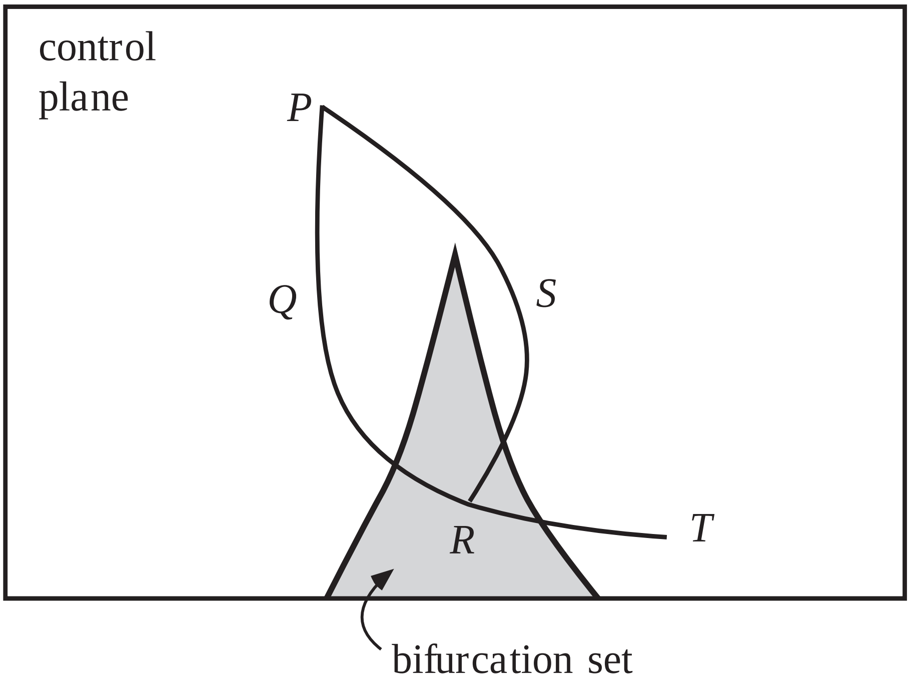
Given a point in the control plane we look above to see what positions are possible on the behaviour surface. We suppose that there is a dynamic flow in space surrounding the behaviour surface, arrows *down* above the surface and up below the surface. In the next diagram we have a drawing of the cross-section *XY*.



The flows in the lines above X_1, X_2, X_3, X_4, X_5, correspond precisely to the flows (1), (2), (3), (4), (5) mentioned earlier. Above X_1 only one position of equilibrium is possible, but as we move the point in the control plane smoothly along the line *XY*, at X_2 another equilibrium point appears, separating into two, giving a total of three equilibrium points over X_3, two stable and the middle unstable. At X_4 the two lower ones coalesce and at X_5 they have been annihilated leaving only the higher one. Thus as we move the control point along the line *XY*, the behaviour point on the surface above is initially on the lower part of the behaviour surface, but at the point X_4 it must jump to the higher surface. This is the "catastrophe". A continuous change in the control variable forces a discontinuous change in behaviour. In fact if we disturb a point off the behaviour surface in any of the vertical lines above a given point in the control plane, we see that the dynamics above X_1 force a return to the only attractor on the behaviour surface. Above X_3, however, a major displacement of a particle on the lower part of the behaviour surface may end up on the higher part, or vice versa. We can consider the picture to represent a control point on the control surface and a behaviour point vertically above constrained to move according to the dynamical flow on the vertical line. The behaviour point is also subjected to perturbations. As we move from *X* to *Y* along the line between X_2

and X_4 the perturbations may cause a jump from one part of the surface to the other. To achieve a jump the perturbation must move the particle beyond the middle part of the surface. For a given size of perturbation this is more likely to occur upwards near X_4 and downwards near X_2. Under such perturbations, moving along XY from X_1 to X_5, a jump is bound to occur upwards at X_4 or before and returning from X_5 to X_1, it must occur downwards at X_2 or before.

Different paths in the control plane may result in widely differing behaviours. For example, paths *PQR* and *PSR* both start and end at the same point but the behaviour in the former ends on the lower surface and in the latter on the higher surface.

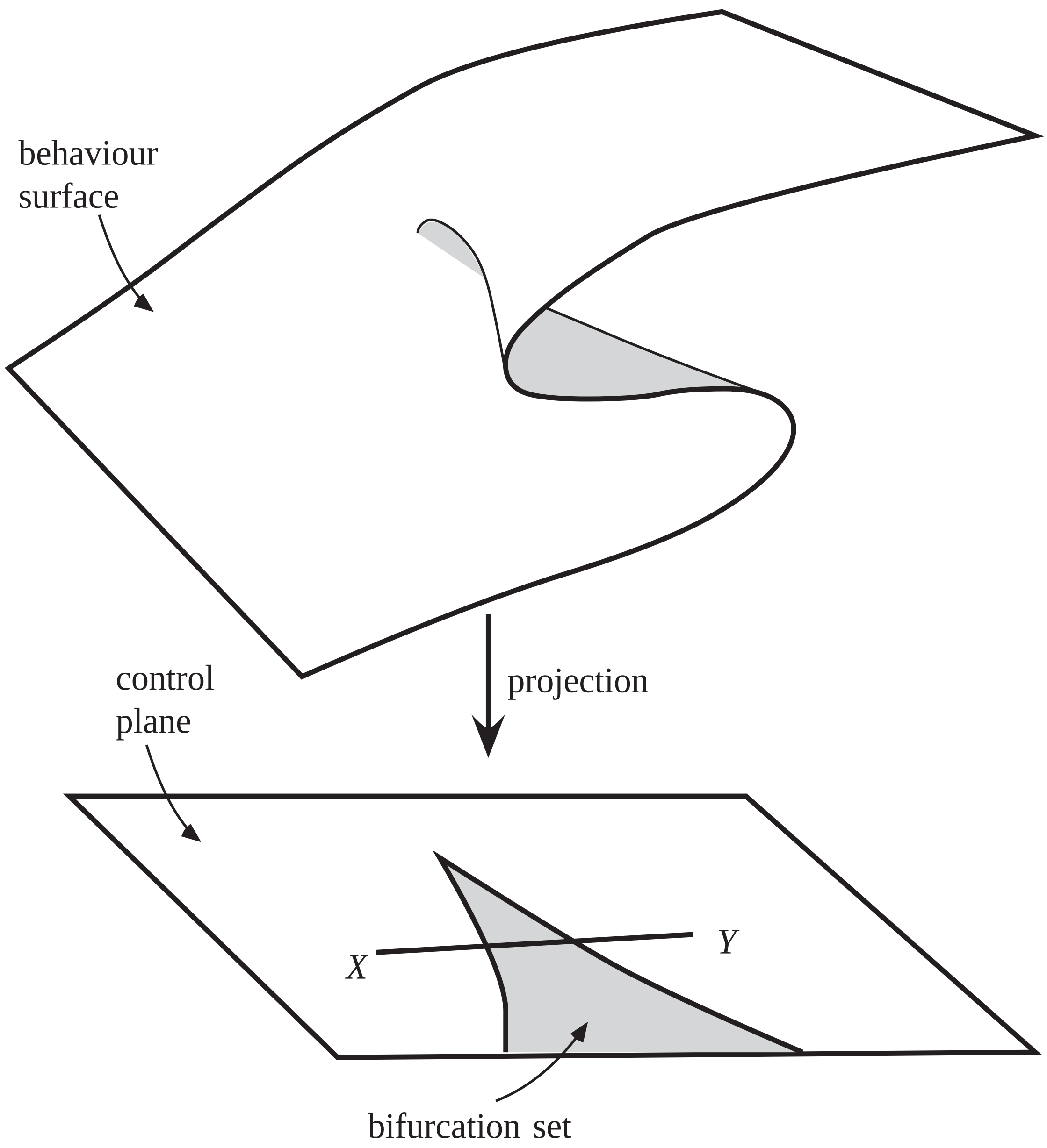


In neither case is there a catastrophe. However, *PQRT* has a catastrophic jump whereas *PSRT* does not. Both end up in the same place on the higher part of the catastrophe surface.

Transition to concrete operations

In this phase the child's visual perception is conflicting with his growing internal conviction. The former attracts him to non-conservation, the latter to conservation. In the control plane we put axes measuring his strength of perception and strength of conviction. The height above in the behaviour space is a measure of his behaviour in the sense that the lower part of the behaviour surface indicates the attractor of non-conservation and the upper part indicates conservation. The far back part of the surface where there is no fold under indicates the preconceptual thought prior to the transition phase.

Strong perceptual stimulus leads to non-conservation, strong inner conviction leads to conservation. A path such as P_1 would represent a typical path and on the segment *DE* strong perturbation could lead to jumps in



behaviour in the transition phase. Path P_2 is still at the non-conservation stage but in transition, path P_3 indicates a path that leads straight to conservation without transition. Are all these paths possible? Are all parts of the surface accessible? Are the axes in the right sort of position? Such a model has various compelling facets. It indicates how in different children the transitions may seem to involve sudden large changes but in others there may be small or even seemingly continuous changes (path P_3?). The essential factor to grasp is the discontinuous change in behaviour together with the conflicting conclusions caused by following different paths. Further interesting examples of this transition may be found in [2].

Incidentally, the theoretical notion of a “grouping” which Piaget uses to describe the logico-mathematical operations have rightly been attacked as being vague and mathematically inconsistent. In the presence of a catastrophe model, however, we may ask whether a logical description of thought in the transition phase is a possibility at all. Certainly such a model would not allow logically contradictory results to follow from given premises just by taking different routes to the conclusion. The presence of contradictory conclusions (conservation and non-conservation) found in the transition phase by Piaget himself suggests that any model of thought in this phase using classical logic is bound to end up with internal contradictions. It may come as no surprise then to find that Piaget's models of concrete operational thought contain such inconsistencies. The catastrophe theory of the cusp tells us that a given path in the control space will lead to a preordained result in the behaviour space (at least when there is no perturbation on the behaviour point occurring).

In the presence of a transition phase we would, therefore, require a mathematical description of logic where the conclusion of a sequence of deductions was preordained, yet different paths could lead to different conclusions, possibly with catastrophic jumps on the way. In a phrase we require *a path-dependent logic*, though no such theory seems available at the moment.

Point nine recurring

As part of a number of questionnaires given to first-year mathematics students at Warwick, one page was devoted to sequences.

The questions included the following:

(i) Have you been taught the concept of the limit of a sequence $s_1, s_1, \dots, s_n$. in school?

A with precise definition
B informally
C not at all.

(Tick the most appropriate.)

(ii) Whether your answer is A, B or C, try to find the following limits (if they exist):

(a) $\lim_{n\to\infty} (\frac{1}{2})^n =$

(b) $\lim_{n\to\infty} \dfrac{n^2}{2n^2+1} =$

(c) $\lim_{n\to\infty} (1 + \frac{9}{10} + \frac{9}{100} + \ldots + \frac{9}{10^n}) =$

(d) $\lim_{n\to\infty} (1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}) =$

(iii) If you know the definition of the limit of a sequence, write it down:

$s_n \to s$ as $n \to \infty$ means:

(iv) Is $0\cdot\dot{9}$ (nought point nine recurring) equal to one, or is it just less than one? Explain the reason behind your answer.

As might be expected, out of 36 volunteers completing the test, only 10 claimed knowing a precise definition and only seven of these gave a suitable definition. The majority (21) claimed to have met the notion informally and of these only one could give a definition. The remaining interest in the questions centred on the answers to part (c) of (ii) and part (iv). Whereas most (29) answered correctly to (ii) (c):

$$\lim_{n\to\infty}\left(1+\tfrac{9}{10}+\tfrac{9}{100}+\ldots+\tfrac{9}{10^n}\right)=2,$$

there were a wide variety of answers to part (iv):

$$0\cdot\dot{9} = ?$$

Fourteen said $0\cdot\dot{9} = 1$, two hedged their bets and twenty said it was just less. Significantly, of the seven who claimed a precise definition and could give one, only one of these said $0\cdot\dot{9} = 1$. Looking at the answers, clearly there was a conflict between 0·99…9 to a finite number of places and the infinite expansion.

This was not the only problem; infinitesimal ideas insinuated themselves, typical responses being:

> "It is just less than one, but the difference between it and one is infinitely small."

> "Just less than one, because even at infinity the number though close to one is still not technically equal to one."

Both of these answers came from students claiming to know a precise definition of a limit. Even students claiming $0\cdot\dot{9} = 1$ were troubled by infinity.

> "I think $0\cdot\dot{9} = 1$ because we could say '$0\cdot\dot{9}$ reaches one at infinity' although infinity does not actually exist, we use this way of thinking in calculus, limits, etc."

The existence of thirteen solutions saying:

$$\lim_{n\to\infty}\left(1+\tfrac{9}{10}+\tfrac{9}{100}+\ldots+\tfrac{9}{10^n}\right)=2$$

but $0\cdot\dot{9} < 1$ was a factor which puzzled me, because on the face of it they were conflicting answers to similar problems. At this stage a catastrophe interpretation had not occurred to me. Then Rolph Schwarzenberger tried a test on the students of stunning simplicity.

In a lecture, without warning, he said:

> "I will write down a number of decimals and ask you to express them as fractions in simplest form. Example: $\cdot 5 = \tfrac{1}{2}$."

He then wrote on the blackboard:

(1) ·25

(2) ·05

(3) ·3

At this stage a member of the audience asked "Is that point three recurring?" to which Rolph replied: "No, not recurring." He then wrote down:

(4) $\cdot\dot{3}$= ·333… and said "this is recurring."

(5) $\cdot\dot{9}$= ·999… and said "recurring".

The results were dramatic. Twenty-four now said $0\cdot\dot{9} = 1$ or $0\cdot\dot{9} = 1/1$. Only one who had previously said $0\cdot\dot{9} = 1$ on the earlier paper did not tackle it. (He was someone with a precise definition of a limit although his justification of $0\cdot\dot{9} = 1$ was at that stage "because there are no possible numbers between".)

Neither of the students who hedged their bets about $0\cdot\dot{9} = 1$ earlier said that $0\cdot\dot{9} = 1$ on the test. One did not attempt it, the other wrote

$0\cdot\dot{3} = 1/3$ (not quite), $0\cdot\dot{9} = 1$ (not quite).

There were 13 students who previously affirmed $0\cdot\dot{9}<1$ who now said $0\cdot\dot{9}=1$. More interesting were the answers which, by the manner in which they were written, exhibited the actual conflict in the mind of the student. Here are the interesting answers to parts (4) and (5) from students who had earlier claimed $0\cdot\dot{9}<1$:

A (4) $0\cdot\dot{3} = \frac{1}{3}$
(5) $0\cdot\dot{9} = 3\times\cdot\dot{3} = 3\times\frac{1}{3}$ = rubbish

B (4) 0·333… = $\frac{1}{3}$ no fraction (this answer was crossed out)
(5) 0·999… = " "

C (4) $0\cdot\dot{3} = \frac{1}{3}$
(5) $0\cdot\dot{9} = 1$ or (none exists)

D (4) $0\cdot\dot{3} = \frac{1}{3}$
(5) $0\cdot\dot{9} = 0\cdot999$

E (4) $0\cdot\dot{3}= \frac{1}{3}$
(5) $0\cdot\dot{9} \approx 1$.

Student A clearly saw the conflict and responded with exasperation. In the case of student B we cannot be sure in what order he wrote the words from the script, but discussion afterwards clearly showed he came to the conflict in question 5 and this forced him back to cross out his answer in question 4. Student C visualised two possible alternatives, but found it difficult to express, D switched from $0\cdot\dot{3}$ as an infinite decimal to $0\cdot\dot{9}$ as a finite decimal whilst E switched from $0\cdot\dot{3}$ as an infinite decimal to $0\cdot\dot{9}$ as a number approximately equal to one.

These illustrate beautifully the path dependence of the decision making process in the area of conflict. The conflicting factors in some cases seem to be whether to think of the decimal as being a finite expression on the one hand or a limit on the other. Since the students mainly had only an informal idea of limit we should not expect them to be able to resolve the conflict. It is clear that the concept of infinity as a limit process and infinitesimals as arbitrarily small numbers also enter into the picture (how strange that these historical difficulties raise their heads in our emancipated modern mathematics!). Repellent forces are acting in many students' minds when infinite processes are involved. The reasons for these are more subtle. One is that the student may expect infinite decimals to be in one-one correspondence with real numbers. This would lead to the interpretation that $0\cdot\dot{9}$ is less than 1. Another is the verbal definition of limit:

$s_n \to s$ as $n \to \infty$ means that as n gets very large, so s_n gets close to s.

The phrase "gets close to" is the source of trouble because in colloquial English this carries the connotation "gets close to, but is not actually coincident with". The definition of a limit of a continuous function is even worse in this respect for when

$$\lim_{x\to a} f(x) = L$$

we must keep x distinct from a whereas $f(x)$ may coincide with L. Many students read into the definition here that $f(x)$ gets close (but not equal) to L. I have resisted applying the cusp catastrophe to this situation (although a simple model could be considered with the strength of finite and infinite decimal concepts as conflicting factors) simply because other repellers may be present in the student's mind caused by conflicts over the notion of infinity.

The important facets to note are the existence of a conflict, the path dependence of decisions and the sudden jumps from one decision to another. It is interesting to note the paths taken by students to see that $0\cdot\dot{9} = 1$. Some saw that $0\cdot\dot{3} = \frac{1}{3}$, so $0\cdot\dot{9} = 3\times\frac{1}{3} = 1$. One said that:

$$10\times 0\cdot\dot{9} = 9\cdot\dot{9} \quad \text{(not sure if valid)}$$

$$\Rightarrow 9\cdot\dot{9} - 0\cdot\dot{9} = 9\times 0\cdot\dot{9}$$

$$\Rightarrow \qquad 9 = 9\times 0\cdot\dot{9}$$

$$\Rightarrow \qquad 1 = 0\cdot\dot{9}$$

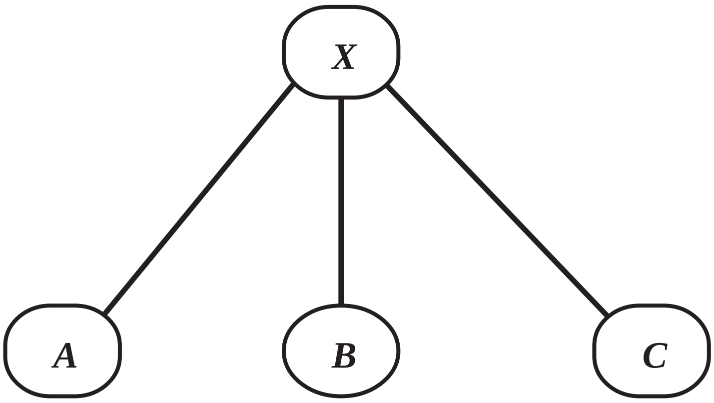
The last word on this should be left to a thirteen-year-old child who was asked about $0 \cdot \dot{9}$ and thought it was less than I and was told it was equal to 1, but given no reason. He went away to school and four weeks later wrote:

> "A question you asked me sometime ago was whether I thought $0 \cdot \dot{9}$ was less or equal to one. I now think and know it is equal to one because if $\frac{a+b}{2} = a$, then $a = b$, so if $a = 0 \cdot \dot{9}$ and $b = 1$, then $a + b = 1 \cdot \dot{9}$ which when divided by two gives you $0 \cdot \dot{9}$, so $0 \cdot \dot{9} = 1$. (I was told this by my maths master.)"

Note that in this solution, division of two into 1·999… by long division gives 0·999…, which is a legal operation in the child's schema. He not only thinks, he *knows.*

Suitability of Models

There is always a danger in using models that they will be misinterpreted. The idea of a schema as a partially ordered lattice is open to such problems. If concept *X* is above concepts *A*, *B*, *C*, it may be interpreted that before one can understand *X* it is essential to understand *A*, *B*, *C*.



On the contrary it may happen that understanding *A*, *B*, then *X* might fit *C* neatly into context. Intelligent learning (for example, using advance organisers [1]) might well proceed in this way. This is the danger in looking for a simple "tree of knowledge" for mathematics. Such a notion may be of positive use to describe what happens in the cognitive development at primary level, but at higher levels it may not be clear which are roots and which are branches. Given an ordered field, we can add completeness either by insisting:

A every non-empty set has a least upper bound,

or

B (i) the field is archimedean;
(ii) every cauchy sequence converges.

Which is a pre-requisite for the other, A or B (i) and (ii)? We can logically deduce B (i) and (ii) from A, so perhaps A is a prerequisite, on the other hand we can take B (i) and (ii) as axioms and then deduce A. The examples of two-way dependence in mathematics are legion, so the tree of concepts need not even be partially ordered. Surely what we are looking for are sensible ways of plotting the curriculum and there must be many such paths.

It is vital that a picture takes into account the repellers which hold concepts and schemas apart as well as attractors which cause linking flows between them. This is not to say that the catastrophe model is perfect because it too may be misinterpreted by a lack of understanding of the qualitative nature of catastrophes and of the variety of other possible catastrophes beside the cusp. What a catastrophe model does, however, is to reinforce the important role of the teacher taking the learner through the curriculum. Explanations may be given to the class as a whole, but the occurrence of conflicts in the mind will be apparent immediately to the sensitive teacher. The simplest manifestations are confusion, annoyance, fear, or just a dull lost look in the eyes! Close investigation may reveal the blockage, sudden catastrophic leaps of thought or even path-dependent decision-making. Then it is the job of the teacher to find the conflict and smooth it out in a suitable manner. It may not be close to the concept, so labouring this may not help. The learner may not even be able to say where it is. The science of teaching lies in the clear exposition of the main ideas, but the art lies in seeking the individual difficulties and removing the conflicts. It is hoped to investigate these ideas further and I would be pleased to hear from anyone who can give details of catastrophe situations in learning.

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