Fundamental Cycles in Learning Algebra: An Analysis

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Over the past 10 years a range of learning theories have appeared in which certain fundamental cycles of learning have been used to describe development of algebraic ideas. In this paper we consider the SOLO Model (Biggs & Collis, 1991) and compare and contrast it with various theories of process-object encapsulation (e.g., Dubinsky, 1991; Gray & Tall, 1994; Sfard, 1995). Our purpose is not to attempt to produce a unified theory but to look at the meanings implicit in each theory and to see where each may shed light on the other, leading to theoretical correspondences and dissonances.

In a range of learning theories there appear certain fundamental cycles of learning that arise in similar form in a wide range of contexts. In this paper we consider two formulations of such ‘universal cycles’ in the development of algebraic knowledge. One is found in the SOLO (Structure of Observed Learning Outcomes) Model. This framework can be considered under the broad descriptor of neo-Piagetian models. It shares much in common with the ideas of such theorists as Case, Fischer, and Halford. In particular the SOLO model comprises a recurring cycle of three levels referred to as *unistructural, multistructural, and relational.* The second formulation concerns various theories of process-object encapsulation, in which processes become interiorised and then conceived as mental concepts, which has been variously described as *action*, *process*, *object* (Dubinsky), *interiorization*, *condensation*, *reification* (Sfard) or *procedure*, *process*, *concept* (Gray & Tall). Our purpose in this paper is not so much to attempt to produce a unified theory incorporating these perspectives. Instead, it is to look at the meanings implicit in each broad theory and to see where each may shed light on the other, leading to theoretical correspondences and dissonances.

It must first be explained that these theories have been developed for different purposes and are also applied in somewhat distinct contexts. We therefore begin by considering each in broad terms.

Universal Cycles in Each Mode in SOLO

The SOLO Model, developed by Biggs and Collis (1982), grew out of a comprehensive analysis of research into Piaget’s developmental theory. It was observed that children simply did not give all responses consistently within the successive stages (sensori-motor, pre-operational, concrete operational, formal operational) proposed by Piaget. So, instead of stages, Biggs and Collis formulated *modes* of thinking, replacing pre-operational by ‘ikonic’ (a visuo-spatial activity supported by growing language usage) and adding an additional mode to give five modes, sensori-motor, ikonic, concrete symbolic, formal, post-formal. Each of these modes comes on-line in an individual in the given sequence but all modes attained remain available for use as appropriate. Thus, it is possible in a given mode to have access to all previous modes if required.

Of particular reference to algebraic thinking are the two modes concrete symbolic and formal. In the concrete symbolic mode a person thinks through use of a symbol system such as written language and number systems. This is the most common mode addressed in learning in the upper primary and secondary school. For students in western cultures this mode becomes available from about the age of five or six years. Significant in this mode, student responses are related to real-world referents. In the formal mode a person considers more abstract concepts. This can be described as working in terms of 'principles' and 'theories'. Students are no longer restricted to a concrete referent. In its more advanced form this mode involves the development of disciplines. Such thinking becomes apparent for high achieving students around 15 or 16 years of age.

Within each of these modes, various types of response were observed which were formulated as a ubiquitous cycle of levels referred to as unistructural, multistructural, and relational. The significance of these levels is that the underlying structure of a student's response could be described, respectively, in terms of a single aspect, a number of independent aspects, and a linking of the aspects previously identified into an integrated whole. We refer to this as a UMR cycle. Biggs and Collis (1982, 1991) theorized that there was one cycle in each mode. The development on from any particular UMR cycle, as determined by a response, was for them, the emergence of a new unistructural level in the next acquired mode. This new unistructural level now becomes the basis of development of a new UMR cycle within the newly acquired mode.

Subsequent empirical study by several writers (e.g., Campbell, Watson & Collis, 1992; Pegg, 1992) revealed various individual conceptual structures going through a UMR cycle. Not only were these to be found singly within each mode but multiple cycles were identified within a given mode. This finding would fit with the natural growth of an evolutionary species: the individual meets new stimuli and begins to react first to one aspect, then another, to give multiple responses, which begin to be related together, then the whole structure is conceptualized as a new single structure. This structure can retain characteristics of the initial cycle or can take on the characteristics of a new mode. In this way, a theoretical perspective that began as a classification of observed learning outcomes returns to its roots as a theoretical model of (local) epistemological development of concepts.

Process-object Encapsulation Cycles in Process-object Theories

Theories of 'process-object encapsulation' were formulated at the outset to describe a sequence of cognitive growth. Each of these theories, founded essentially on the ideas of Piaget, saw cognitive growth through actions on existing objects that become interiorized into processes and then encapsulated as mental objects.

Dubinsky described this cycle as part of his APOS theory (action-process-object-schema), although he later asserted that objects could also be formed by encapsulation of schemas as well as encapsulation of processes. Sfard (1991) proposed an operational growth through a cycle she termed interiorization-condensation-reification, which she complemented by a 'structural' growth that focuses on the properties of the reified objects formed in an operational cycle.

Gray and Tall (1994), focused more on the role of *symbols* acting as a pivot, switching from a process (such as addition of two numbers, say 3+4) to a concept (the sum 3+4, which is 7). The entity formed by a symbol and its pivotal link to *pro*cess or con*cept* they named a *procept*. They observed that the growth of procepts occurred often (but not always) through a sequence which they termed procedure-process-procept. In this model a procedure is a sequence of steps carried out by the individual, a process is where a number of procedures (≥0) giving the same input-output are regarded as the same process, and the symbol shared by both becomes flexibly used to evoke process or concept.

The various process-object theories have a spectrum of development from process to object summarized in the following table (Tall et al. (1999), reproduced with permission.)

	Process	…	Object
Piaget (50s)	action(s), operation(s) …	…	thematized object of thought.
Dienes (60s)	predicate …	…	subject.
Davis (80s)	visually moderated sequence … *each step prompts the next*	integrated sequence … *seen as a whole, and can be broken into sub-sequences*	a thing, an entity, a noun.
Greeno (80s)	procedure …	input to another procedure …	conceptual entity.
Dubinsky (80s)	action … *each step triggers the next*	interiorized process … *with conscious control*	encapsulated object.
Sfard (80s)	interiorized process … *process performed*	condensed process … *self-contained*	reified object.
Gray & Tall (90s)	procedure … *specific algorithm*	process … *conceived as a whole irrespective of algorithm*	procept. *symbol evoking process or concept*

Table 1: The transition between process and object

The process-object theories of Dubinsky and Sfard were mainly based on experiences of students doing more advanced mathematical thinking in late secondary school and at university. For this reason their emphasis is on formal development rather on earlier acquired forms of thinking such as associated with Piaget's sensori-motor or pre-operational stages.. Note too that, as represented here, Sfard's first state is referred to as an 'interiorized process', which is the same name given in Dubinsky's second, however, both see the same main components of the second stage:– that the process is seen as a whole without needing to perform the individual steps.

The work of Gray and Tall, focusing on symbols in mathematics, is more appropriate to arithmetic and algebra than to geometry, where they see an object-based theoretical development consonant with the work of van Hiele and Sfard. However, they see the change to axiomatic mathematics (incorporated in say van Hiele level 5 and SOLO formal/post-formal) a significant shift of meaning. Here selected properties (axioms) are manipulated logically in a structural Bourbaki sense to build a theory of axioms and theorems (Tall, 1995).

In this paper we concentrate on the procedure-process-procept cycle found typically in arithmetic, algebra and symbolic calculus, which has its analogies with Dubinsky's APOS and Sfard's operational mathematics. The work of Gray and Tall has two distinct levels of study — a specific analysis of the growth of whole number arithmetical concepts in children that enabled them to use arithmetic symbols as procepts, and the general development of the theory of symbols as procepts in other areas of mathematics (principally in arithmetic, algebra and calculus). Whilst the first of these, looking at the development of whole number arithmetic, sees a distinct procedure-process-procept cycle in which the procept (of whole number) becomes a given in building the procept of sum and then of product, the comparison of this growth with those of other procepts reveals differences in understanding.

The procept of addition/sum of whole numbers

Adding two numbers together begins with counting two sets, putting them together, then counting them all. This gives a procedure (called 'count-all') which requires physical objects for counting and involves three counting procedures using number-sequences starting with 'one'. This is successively compressed into "count-both" (two counting procedures, the second carrying on from the first). "count-on" (one counting procedure) starting after the first number and carrying out a single counting procedure. Children also remember various 'known facts' whatever level they have reached (including children who are still at the "count-all" stage). Some children begin to "derive facts" eg "8+4" is "8+2+2" which is "12". However, Gray and Tall found no children in their sample who sometimes used "count-all" who were able to "derive facts". Even children who can "count-on" may solve a problem such as "8+4' by counting-on "9, 10, 1l, 12" (perhaps referring to four fingers for the count) to get the answer "12", yet not remember the original problem "8+4". Such a child can "do" 8+4 by counting on, yet not "know" what "8+4" is. Learning to add two numbers can therefore involve several different procedures such as count-all, count-both, count-on, count-on-from-larger (working out 2+9 by counting-on 2 after 9). Realising that all these different procedures gives the same process moves the individual on towards the concept of addition as a process, but it requires the flexible use of derived facts by decomposing and recomposing numbers in sensible ways that leads the flexible *procept* of sum.

Development of other procepts

There are many other procepts in mathematics. These include sum, product, power, negative number, fraction, ratio/rate, trigonometric ratio/function, algebraic expression, function, limit of a function, limit of a sequence, limit of a series, derivative, and integral.

In different contexts, Tall et al. (2000) found that the nature of previous experience of the learner and the specific nature of the new concepts encountered played a significant, even overwhelming, role, so that there were distinctly different problems for the growing individual to face. Whole number arithmetic has *operational* procedures to give a specific answer, which already happened to be a whole number. Fractional arithmetic demanded the construction of a new entity—a fraction. Algebra gave a new problem in that symbols such as $2+3x$ involved a different kind of entity that is only *potentially operational*, in the sense that they could not be evaluated until x was given a numerical value. So children faced the problem of manipulating symbols which they might think of as processes they could not evaluate. Later still, in the limit concept, there is a process which is *potentially infinite* and may have no finite procedure of evaluation. In the calculus, the growth from potentially infinite process to limit concept is a huge step for students to take.

The cognitive construction of each of these can be compared with a UMRU sequences of SOLO. More subtly, the construction of these concepts can also be considered in different SOLO modes. For instance, the evidence from a wide range of studies shows that almost *no* student has a coherent understanding of limit at the first meeting (Cornu, 1991). Tall (1985) decided that it was therefore inappropriate to begin the calculus sequence with the limit concept. To do this, he approached the idea in a more primitive visuo-spatial mode, building on dynamic ideas rooted in the sensori-motor and ikonic modes of thought. This involved using the idea of 'local straightness' to "see" the slope of a curve through high magnification. In this approach the student could look along the graph of a function and see the changing slope as a function in itself. By working in a more primitive visuo-spatial mode, the students could be readily acquainted with aspects of the formal theory in a physically meaningful way prior to a formal development. Tall (1985) showed how a locally straight approach linked simultaneously to symbolic simplifications of the slope function could give a fundamentally human basis to an otherwise difficult formal theory.

In each of these conceptual development patterns there are differences in the development of the procedure-process-procept sequence. The number procept has a significant pre-procedure activity to develop the procedure of counting. Assuming the procept of number has been constructed, the procept of sum has a sequence of more efficient procedures until there is the possibility of flexible approaches to addition using the sum as a procept by manipulating known facts to produce derived facts. Of course, if the procept of number is not available before addition is introduced (and it often is not), then there are extra procedural difficulties in handling addition (such as complicated manoeuvres. to count on fingers). These can lead to a bifurcation between those who remain fixed in counting procedures and those who develop flexible approaches to arithmetic that involve less cognitive strain and therefore greater possibility for future developments.

Algebraic examples using SOLO

A closer study of the development involved using SOLO model reveals the potential for a finer grain analysis than was apparent with the previous models. Consider two algebraic examples. The first item was used initially by Biggs and Collis (1982). The data presented here was when the item was applied to a number of secondary classes in 2001. The item requires students to find the value of x in the following equation

$$(72 \div 36)9 = (72 \times 9) \div (x \times 9)$$

The following responses were identified in the first UMR cycle in the concrete symbolic mode:

Unistructural 1: Student is not sure what the question is asking for and looks for visual clues. They focus on a single relevant feature. “Has it got something to do with the 9s?”

Multistructural 1: Similar in structure to the previous response, however, the student mentions more than one aspect. There is no attempt to link the aspects. “It’s got 9s and 72s on both sides.”

Relational 1: Students take an ‘educated’ guess that takes in to account all the data available. The response still has a visual content and is likely to be incorrect. The student may indicate concern about the difficulty or Tedium associated with the calculations and request a calculator. “36” – possibly because there needs to be a 36 on both sides”

The following responses were identified in the second UMR cycle in the concrete symbolic mode.

Unistructural 2: Calculation of one numerical aspect. Attempts to start the problem but only completes one operation, usually the first available. Calculates one side of the equation or one operation. “72÷36=2”

Multistructural 2: The operations are attempted sequentially. No calculations are used, i.e., does not cancel the 9’s. Often incorrect manipulations are evident.

Relational 2: Students see number patterns in the question. They are able to reframe from direct calculation by carrying out some simplification first. Responses here are likely to be correct.

Formal mode responses: In this case students have a clear overview of what the question is asking. Numerical calculations are used as a last resort. Students can rearrange the order of the symbols to better suit their purposes. Usually skip operations steps.

$$\frac{a}{b} \times y = \frac{a \times y}{b}$$

$$\frac{72}{36} \times 9 = \frac{72 \times 9}{x \times 9}$$

$$\frac{72 \times 9}{36} = \frac{72 \times 9}{x \times 9}$$

Number Patterns

The second example is referred to as the *tiles around square swimming pools problem* (Pegg, 1992). Students were presented with six squares (with side length starting at one unit and the largest having sides of 6 units) which represented ‘swimming pools’. The purpose of the activity was to find a rule for the number of tiles around any given square pool. The tiles were themselves squares of sides one unit.

Responses in the concrete symbolic mode in the first UMR cycle were:

Unistructural 1: Counts the tiles around one pool. “Well number one square is 8 tiles well you count the the rest of the squers and ad them up.”

“There are 8 tiles around the pool”

“Each tile hand numbers to get 8 altogether”

“You get on lot of tiles to go completly around the hole pool. The first one is 8 tiles go around it.”

Multistructural 1: *Counts the tiles around more than one pool and usually tries to tabulate the data.* “1. 8, 2. 12, 3. 16, 4. 20, 5. 24, 6. 28”

“1 pool = 8 tiles; 4 pools = 12 tiles; 9 pools = 16 tiles…..”

and the student numbers the drawn tiles around all the pools showing the number of tiles needed. Note the “1 pool, 4 pools, 9 pools” refers to the pools of side 1, 2 and 3 units, respectively.

“The first pool is the small one the tiles needed is 8. The second one is the oppisit it uses 12 tiles….”

Relational 1: Sees a relationship between the pools, but only in terms of the pools and not towards a general rule. “because there’s four between each one.”

“The rule for the number of tiles surrounding the pool is every one goes 4.”

Lists the tiles for each pool then “there is 4 more tiles in each one.”

“Just add four more tiles to each swimming pool than the previous one had.”

“Each pool needs 4 more tiles. Number of tiles equals square pools plus 8.”

Responses in the second UMR cycle in the concrete symbolic mode were:

Unistructural 2: Sees a relationship between a property of the pools and subsequent number of tiles. The response builds on the ‘four more tiles feature’ that was significant in the Relational 1 response. “Find the perimeter in metres and add 4.”

“Add 4 onto the perimeter.”

“Add 4 tiles to each set of tiles on the edge of the pool.

“Perimeter + 4.”

“The rule is the perimeter times 4.” Here the structure of the response is the same as other answers at this level except the student has chosen the incorrect operation.

Multistructural 2: Students attempt to go further than the previous level by using a more basic and useful element than perimeter to establish a general rule. Here they using side lengths. The rules are not particularly succinct and can be quite convoluted. The rules have a sequential feel to their development.

“Count the number of tiles along one side plus 2 and multiply it by 4, then minus four.”

“The length + 1 x 2 + Bredth + 1 x 2 – 4 = number of tiles.”

“Take the length of the side and times that by two. Then take the length of the side again and add two, then times that number by two and add the two sums together.”

“The length of one side of the pool multiplied by four and then add four to the previous answer.”

“Find the height of the poole then times it by the other hight then times it by 4.”

Relational 2: Students provide a rule which is succinct using minimum operations and using side as variable.

“The amount of squares around one side (inside) multiplied by 4 the add 4.”

“Multiply the side length by 4 and add 4 more tiles in the same units and size.”

“The number of tiles equals the length of the pool, times by four, then adding four.”

Formal mode responses were not provided in this question. Nevertheless, one could hypothesise that it could concern the development of alternative rules formed by manipulating the rule identified at relational 2 algebraically to form new equations.

In both the examples provided the evidence of two UMR cycles was apparent. In both cases the first cycle had a strong visual feature about it as the students developed the appropriate underlying concept. This concept then became a consolidated new unit that became the building block for the second cycle. During the second cycle the same structural development was noted except it was of a form that was more mathematical. However there was still a ‘concreteness’ about the procedures. This cycle concluded by students being able to obtain the correct response successfully. The next level is in the formal mode and it encapsulates the previous UMR cycle in more abstract way. Here the emphasis is removed from the actual real-world elements to the relationships between those elements.

Final Thoughts

The different mathematical contexts almost always involved the spectrum of development through process to concept that is essentially common to all process-object theories. By seeing the multi-procedure level as being distinct from the process level, this gives an apparently unassailable link in surface structure between the procept cycle and the UMR cycles in the SOLO model as follows:

Process-Object Theory	SOLO Model
procedure	unistructural
multi-procedure	multistructural
process	relational
procept	unistructural (new cycle)

However, such an identification would be premature before looking more closely at the microstructure of the learning development. With the conceptual development of various procepts, there are very different conceptual constructions to make. Let us consider one of these.

The procept of number

In the development of the procept of number, the activity of counting (accurately) may be taken as a *procedure*. Counting a set in different orders may be a *multi-procedure* level, then the realisation that different counting sequences for the same set gives the same result may be considered the *process* level. This is what is often called the *conservation of number*. It is but a short step to the *procept* conception of number where the number symbol acts as a pivot between the concept of number and the process of counting.

This long-term development from counting to number concept can also be seen as a compression of brain activity. First there is a highly complex sequential count having to keep control of the whole set to make sure that each element is counted once and once only. As the algorithm gets more fluent, the organisation of pointing at successive elements in a perceived pattern becomes more easy and imposes less cognitive strain. Then the verbal statement accompanying the count, saying "there are one, two, three, *four*' becomes a silent count of the initial items "there are [one, two, three,] four", spoken as "there are …four" with a slight pause for the count, until the child can say "there are *four*", now enunciating the number four as a concept in its own right rather than the last word in a counting sequence.

So far, so good. We have seemingly made a correspondence between process-object stages and SOLO levels. But we are not yet finished. We have not considered the earlier activities that take place to build up to the accurate procedure of counting. There is here a *pre-procedural* level that can take up to a year or more to complete and has itself several sub-stages.

Here SOLO offers of the potential to provide a way forward. The development identified fits comfortably within a UMR cycle in the concrete symbolic mode. Descriptors for development earlier than described here is also possible using SOLO. This is achieved by considering the structure and elements of an earlier cycle within the same mode. In this earlier cycle students can perform simple additions with familiar numbers. This would represent unistructural categorisation. At the multistructural level more than one operation can be performed in sequence provided the context and numbers are familiar. Simple money questions fall into this category. Finally, slightly more difficult procedures, such as giving change with money, is possible at relational 1. However, the techniques employed by the student would have a strong visual element to them, with the student actually visualising, in some form, the money being transacted. It is not until the next level the unistructural 2 level that students are into the mathematics game where any single digit numbers can be used with the four operations. This marks the beginning of school arithmetic as it is generally conceived.

Overall. in each process-object theory, a pattern is observed in a sequence from individual steps to coherent step-by-step sequences to processes conceived as a whole and then to entities that can themselves be manipulated. We will term this sequence SAPO (Steps, Algorithm(s) performed step-by-step, Process conceived as a whole, Object that can be manipulated). We emphasise that this is a *categorisation* of the development strategy rather than a discrete classification, in that it may not always be possible to distinguish precisely in which category a response is placed. However, it has strong links across to the UMRU sequence of SOLO.

The strength in the application of SOLO in its multiple-cycle form is that it not only provides a basis to explore how basic concepts are acquired. It provides us with a description of how students react to reality as it presents itself to them. The second cycle then offers the type of development in the concrete symbolic mode that is a major aim of primary and secondary education. More importantly, the SOLO model operationalises the way humans acquire skills and then consolidate that knowledge to become a tool for further learning or problem solving.

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