Flexible Thinking, Consistency, and Stability of Responses: A Study of Divergence

Mercedes McGowen

William Rainey Harper College
Palatine, Illinois 60067-7398, USA
email: mmcgowen@harper.cc.il.us

David Tall

Mathematics Education Research Centre
University of Warwick, UK
e-mail: david.tall@warwick.ac.uk

The purpose of this study was to investigate whether students classified as 'developmental' or 'remedial' could, with a suitable curriculum, demonstrate improved capabilities in dealing flexibly and consistently with ambiguous notation using various representations of functions. The stability of their responses over time was also a subject of study. The inability to think flexibly leads to a fragmentation in students' strategies and a resulting divergence that is both quantitative and qualitative, between those who succeed and those who do not.

The ability to interpret ambiguous notation and use various representational forms of functions are considered indicators of the ability to think flexibly (Krutetskii, 1969a; Dubrovina, 1992; Shapiro, 1992; Gray and Tall, 1994). Flexible thinking has been characterized in different ways. Krutetskii (1969b) and Shapiro (1992) characterize flexible thinking as *reversibility*, i.e., the establishment of two-way relationships indicated by an ability to "make the transition from a 'direct' association to its corresponding 'reverse' association" (Krutetskii, 1969b, p. 50). Gray and Tall (1994) characterize flexible thinking in terms of *the ability to think proceptually*, i.e., to move flexibly between interpreting notation as a process to do something (procedural) and as an object *to think with* and *to think about* (conceptual), depending upon the context.

In this study, flexibility of thought encompasses both the Krutetskiian and also the Gray and Tall notions, as facets of a broader characterization. In addition to *proceptual* thinking, focused on the use of symbols to evoke both mental process and mental object, we will be interested also in the wider connections between various representations (including tables, graphs and symbols) which we will refer to as *conceptual*. In this sense, the inability to use symbols flexibly causes the *proceptual divide* (Gray & Tall, 1994) is actually part of a broader *conceptual divide*, in which the inability to use symbols flexibly is compounded by the inability to use and translate flexibly among various representational forms.

The subjects of study were twenty-six students enrolled at a suburban community college in the Intermediate Algebra course. A reform curriculum was used, with a process-oriented functional approach that integrated the use of graphing calculator technology. Quantitative and qualitative data were collected throughout the semester on all twenty-six students, from student work, written surveys, pre- and post-test questionnaires, task-based interviews and concept maps. Questions similar to those on the pre- and post-test questionnaires were included in unit exams, and in both the open response and multiple choice portions of the final exam, to allow comparisons among

Completed January 5th 2001; submitted to PME 25.

instruments, question formats and contexts, and consistency of performance and strategy by individual students over time.

Results of pre-and post-test questionnaires, together with results of the open-response final exam and departmental multiple-choice final exam were used to rank the students. Those categorized as most successful were the top fifteen percent of the ranked students and those categorized as least successful were those ranked in the bottom fifteen percent of the class at the end of the semester. These two populations were compared to seek differences between the performances of the most successful and least successful students in the study. Follow-up interviews and analysis of their strategies and concept maps were used to develop profiles of the students in each of these two subgroups. The accumulated data were interpreted within a multi-dimensional framework based on cognitive, sociocultural, and biological theories of conceptual development, using selected insights representative of the overall results of the broad data collected in this study. In an effort to minimize the extent of researcher inferences concerning cognitive processes and to support the validity of the findings, several types of triangulation were used, including data, method, and theoretical triangulation (Bannister *et* al., 1996, p. 147). Profiles of the students characterized as most and least successful were developed based on analysis and interpretation of the triangulated data.

Stability of Students' Responses

Investigations, assignments, small-group work, and class discussions during the semester offered students opportunities for reflective practice. Problems structured similarly to those students had difficulty with at the beginning of the semester were included on assignments, as well as on various forms of evaluation. The responses for the tasks of evaluating a quadratic function with a negative input, [Q6: Given $f(x) = x^2 - 5x + 3$, find $f(-3)$] and an algebraic input [Q7: Given $f(x) = x^2 - 5x + 3$, find $f(t–2)$], reveal the length of time needed to reconstruct inappropriate concept images by the most and least successful students. None of the students classified as 'more successful' or 'less successful' answered Q6 or Q7 correctly on the pre-test. All students in both groups of extremes attempted these problems on the post test. Similarly structured problems were included on a weekly journal assignment (week 3); the first unit exam (week 6); the post-test (week 15) and the open-response final exam (week 16).

Responses for each of the two questions indicate three distinct time periods during which succesful solutions were constructed by different groups of students. The 'most successful' four could solve the problems from week 3 onward (indicated by a dotted rectangle), suggesting they were able to do the problem soon after encountering the theory. This evidence is consistent with the notion that these students (1–4) required almost no practice and very little time to construct an appropriate schema which remained stable throughout the semester. Average students (5–21) generally needed additional practice and experience, with most being successful by week 6. The least successful students (22–26), despite an entire semester of experience and practice, still struggled and were inconsistent in week 16 at the end of the semester.

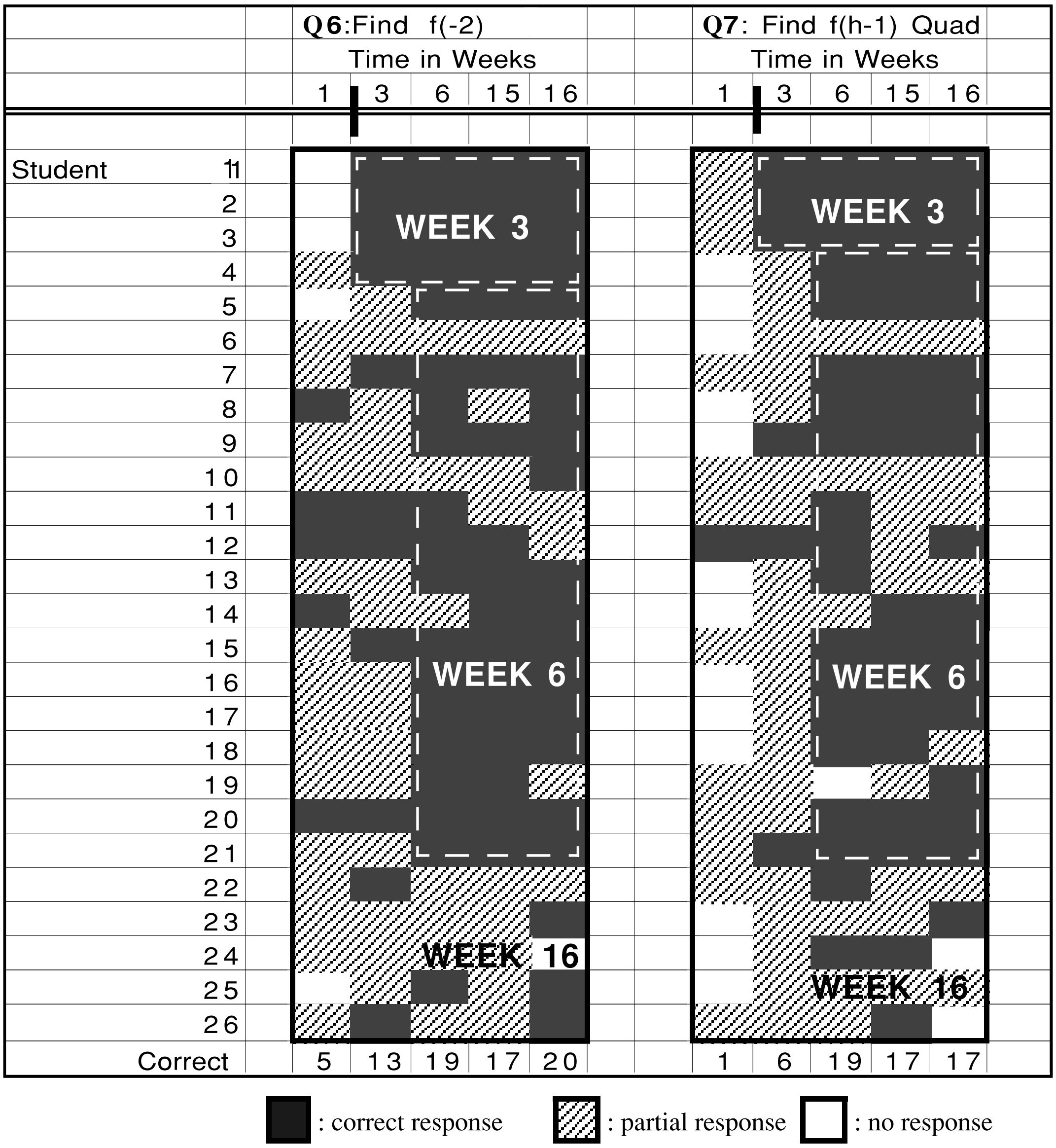


FIGURE 1: Stability of Student Responses

Figure 2 is extracted from figure 1 and simply shows the results of the most successful and least successful. Reading the chart row by row, from left to right, note that both students 24 and 26 correctly answer Q6 and Q7 on the final exam, the post-test question they answered incorrectly the week previously. Two facts should be taken into account in trying to interpret these changed results. First, students 24 and 26, in interviews following the post-test and prior to the final exam, investigated similarly structured problems and reflected on their incorrect post-test responses. Second, the final exam was a multiple choice exam—the post-test was an open response format. It remains an open question as to whether these students have restructured their schemas into more appropriate, stable cognitive collages, or whether their responses are the result of procedures being retained in memory for a brief period of time.

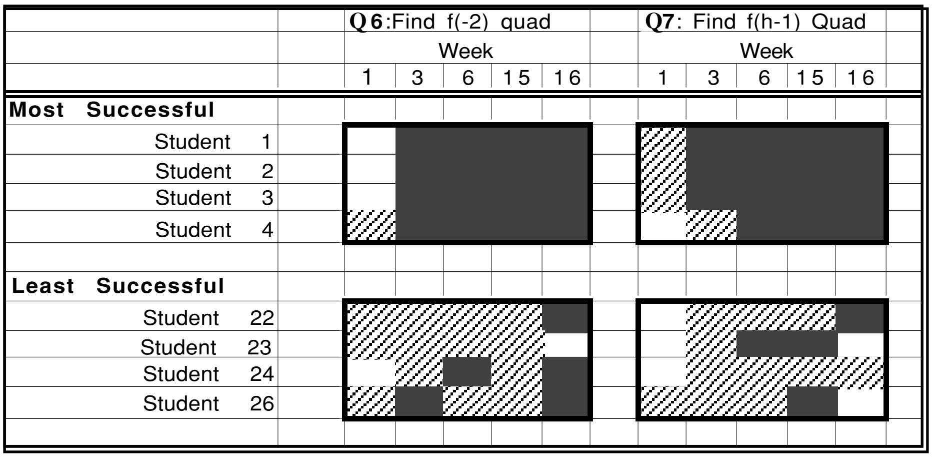


FIGURE 2: Construction of an appropriate Schema

Initially, students who were eventually categorized as most successful demonstrated the same level of competence on Q6 and Q7 as did those who were eventually classified as least successful (Figure 3). Post-test results illustrate yet again the divergence that occurred during the semester. Note that Students 24 and 26 answered Q7 correctly, though their responses to Q6 were incorrect on both pre- and post-tests.

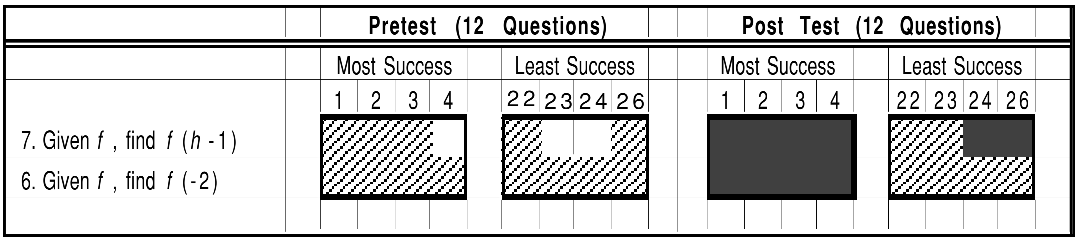
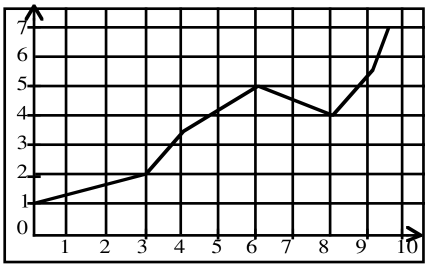


FIGURE 3: Evaluating a Quadratic Function

Neither student was able to answer the arithmetic questions, Q1 and Q3, correctly, suggesting that difficulties interpreting the minus symbol are an underlying cause of the difficulties they experienced in answering Q6.

Ability to Think Flexibly

Pre- and post-test responses document the ability of students in each group of extremes to think flexibly and to switch their train of thought from a direct process to its reverse process. In this analysis we include problems on the reversal of the process as a whole (e.g. Q13 and Q14 (functional symbolic), used on the post-test only. Q13: Given a function f, what is the meaning of $-f(x)$? and Q14: Given a function f, what is the meaning of $f(-x)$? We also focus on the reversal of the *steps* in a process (eg Q10 and Q11(numerical-table); Comparing Q11: $g(f(x))$ with Q10: $f(g(x))$. The latter is of particular importance in the student's interpretation of the meaning of certain notation such as Q1: $-x^2$, which usually means $-(x^2)$ but occasionally may be interpreted as (Q3): $(-x)^2$.. This notion of reversal is relevant in student interpretation of symbolism and the sequence in which the operations are performed. Data indicative of students' abilities to reverse processes using a graphical representation are provided by the pre- and post-test question pairs: Q8 and Q9 (graphical): Given the graph



Q8:Indicate what y(8) = ______

Q9:If *y* (*x*) = 2, what is *x*? ____

The pre-test responses of each group of extremes for the paired reverse process questions are shown on the left in Figure 4. The post-test responses are shown on the right in this table.

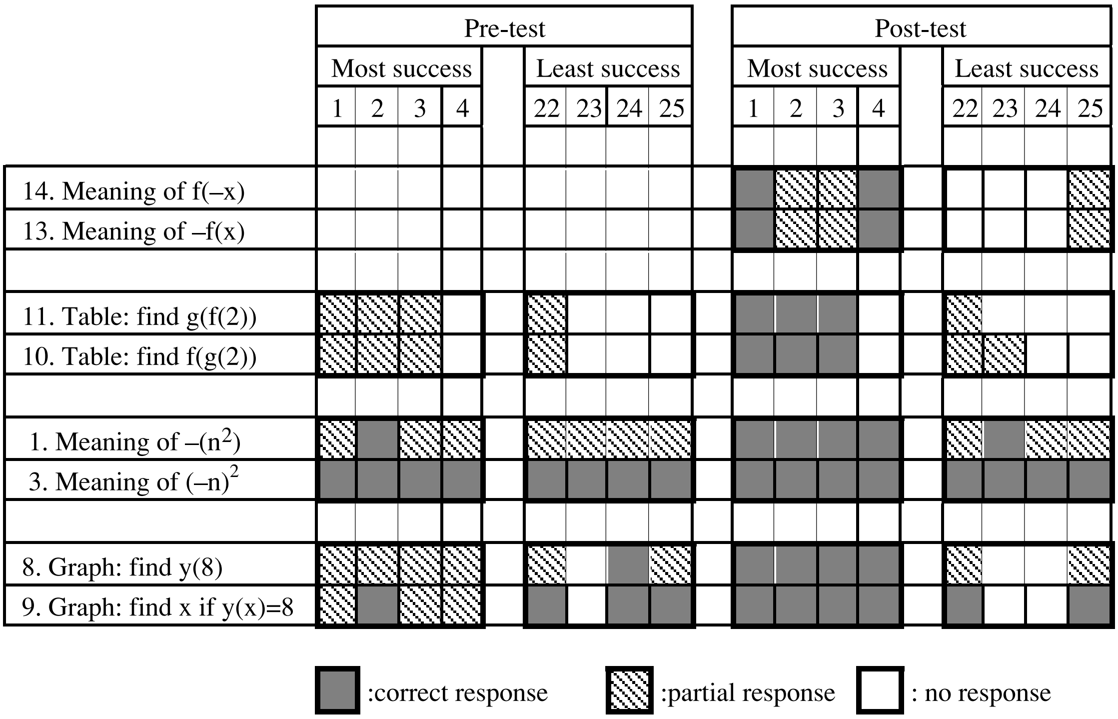


FIGURE 4: Pre- and Post-test Results: Reversal of a Direct Process (or its sequence of steps)

A comparison of pre-test responses in Figure 4 indicate that both groups of extremes demonstrate a similar lack of competence to make sense of either type of reversal at the beginning of the semester, even at the computational level (Q3 and Q1). The four most successful students were all able to perform reversal correctly, when confronted with procedural questions. Reversal questions that were conceptual in the sense that they did not require a specific procedure to determine the answer (such as Post-test Question pairs 10 and 11; 13 and 14) proved more difficult, even for the most successful students. Perhaps this is because conceptual reversal questions frequently require *two* types of flexible thinking— the ability to reverse one's train of thought and the ability to think proceptually— recognizing when part of the symbol or expression indicates a procedure *to do* and when it requires an object *to think about.*

The most successful students were able to find output given an input and find an input given an output. When asked what the expressions $-f(x)$ and $f(-x)$ meant to them, each student saw $f(x)$ as output and x as the input. The difficulty in this problem lay not in their ability to switch their train of thought, but in their ability to interpret the role of the minus symbol in the given context, taking into consideration when it applies in the domain or range.

Despite overall improvement, two of the four most successful students continued to interpret a minus sign in front of a variable to mean that the "value of the output is negative" in Q13, and that the "value of the input is negative" in Q14, even at the end of the semester. *All* of the least successful students interpreted the minus sign in front of a variable to mean that the value is a negative number. In interview this perception was confirmed. Just as a minus sign is placed in front of a number such as 2 to indicate that –2 is negative, so a minus sign in the expression "–*x*" says that the quantity is negative.

Post-test responses of least successful students compared with their pre-test responses, indicate that no improvement in their abilities to reverse a process occurred during the semester. The inflexibility of their thinking extends even to arithmetic computational processes. The ability to reverse their train of thought appeared frozen, regardless of which representation was used. When one recalls that these are undergraduate students in a class where graphing calculators were an integral component of the course which they were encouraged to use, not only in class but on all assessments, these results are even more discouraging.

By the end of the semester the divergence that has occurred is pronounced The most successful students demonstrated that given the graph of a function, they were able to evaluate a function and to reverse thinking to solve for a specified value. They extended their knowledge about function evaluation to include evaluating a composition of a function using tables and were able to reverse their train of thought, first using the output of a function, *g*, as input in another function, *f*, then using the output of *f* as input into the function, *g*. They were also able to square a negative number as well as find the additive inverse of a number squared. In addition, the most successful students were able to describe a direct process and its reversal, distinguishing them as distinct processes.

The students who were least successful were able to answer procedural questions involving a reversal of process and functional notation using a graph (P8 and P9) to a greater extent than they were able to answer questions using a table or questions that did not require a procedure in order to answer the question (P10 and P11; P13 and P14). Procedural questions which involved looking up a value in a table to evaluate a function without the rule being given proved to be the most difficult for these students, confirming earlier research that describes students' inability to deal with functions unless the function rule is give explicitly. Responses to post-test and final exam questions related to reversing a direct process, similarly structured but using different formats (post-open response; final exam, multiple choice) and various representations, are shown in Figure 6. The most successful students demonstrated flexibility in their ability to use various representations, alternative procedures, and effective use of the graphing calculator. They demonstrated the ability to translate among representations, switching from pencil and paper to the graphing calculator and back again freely and comfortably.

The least successful students usually selected and used only one representational form to investigate and solve a problem—even when an alternative would have been more efficient or appropriate. Their choice was invariably the more familiar symbolic procedure—an indication of, and reaction to, cognitive stress. It was as if they were saying "I can't deal with all of this! I'll deal only with this piece!" as they struggled to master new material. The SOLO taxonomy (Structure of Observed learning outcomes) involves a hierarchy of "pre-structural, unistructural, multistructural, relational, extended abstract" levels to evaluate the quality of learning in many subject areas (Biggs & Collis, 1982, p. 25). The response, "I can't deal with all of this, I'll only deal with this piece" suggests a classification at the unistructural or, possibly, the (unconnected) multistructural level, of the SOLO taxonomy.

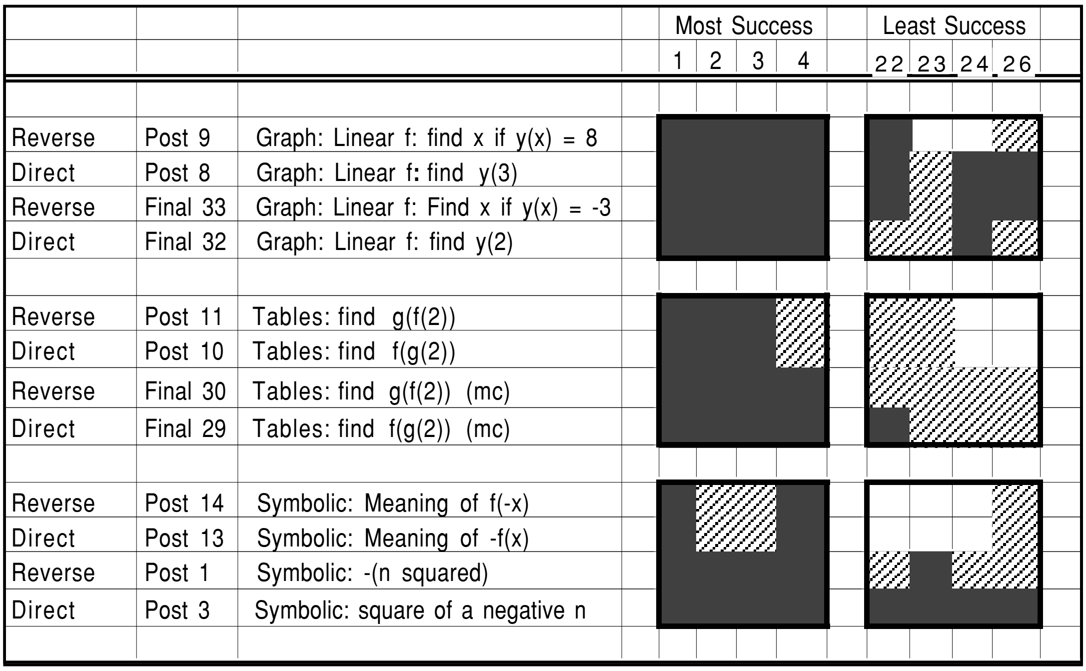


FIGURE 6: Post-test & Final Exam: Reversal of a Direct Process

The contrast between successful students' reconstruction of inappropriate procedures in a relatively short time period and the failure of the least successful to do so even after sixteen weeks is striking. The two groups of students entered a course with no apparent differences in competence and skills in the initial tests, yet, in a relatively short period of time, divergence set in. Successful students developed mastery of the two procedures involving evaluation of quadratic functions by the third week of the semester, maintaining consistency in their responses throughout the semester.

The least successful students were not only unable to develop mastery but, despite many opportunities for reflective practice, were unable to develop any degree of proficiency on material they had already seen in previous mathematics courses by the end of a sixteen week semester. They were somewhat more able to deal flexibly with procedural questions involving ambiguous functional notation than they were with traditionally formatted questions. These students performed best on questions that involved procedural tasks such as the evaluation of linear functions or determining algebraic representations of linear functions from graphs in a multiple choice format. They demonstrated no consistency of performance across different formats, contexts or on procedural activities in which the function rule was not stated, such as evaluating compositions of functions.

The findings of this study indicate a qualified response to the question of whether students classified as 'developmental' and/or 'remedial,' could, with suitable curriculum demonstrate improved capabilities in dealing flexibly and consistently with ambiguous notation and various representations of functions. For some students the answer is "Most definitely!" For other students, the search for activities that will break down the barriers of inflexible thinking and negative attitudes continues.

References

Banister, P., Burman, E., Parker, I., Taylor, M., & Tindall, C. (1996). *Qualitative Methods in Psychology: A Research Guide*. Buckingham: Open University Press.

Biggs, J. B. & Collis, K. F. (1982). *Evaluating the Quality of Learning: The SOLO Taxonomy*. New York, NY: Academic Press.

Davis, Robert B. (1984). *Learning Mathematics: The Cognitive Science Approach to Mathematics Education*. London: Croom Helm Ltd.

Dubrovina, I.V. (1992). The Nature of Abilities in the Primary School Child. In Kilpatrick, J. & Wirszup, I. (Eds.). *Soviet Studies in the Psychology of Learning and Teaching Mathematics*. Chicago, IL: University of Chicago Press.VIII, 65–96.

Gray, E.M. & Tall, D. O. (1994). Duality, Ambiguity, and Flexibility: A "Proceptual" View of Simple Arithmetic. In *Journal for Research in Mathematics Education*, 25, 2, 116–140.

Krutetskii, V.A. (1969a). Mathematical Aptitudes. In Kilpatrick, J. & Wirszup, I. (Eds.). *Soviet Studies in the Psychology of Learning and Teaching Mathematics*. Chicago, IL: University of Chicago Press. II, 113–128.

Krutetskii, V.A. (1969b). An Investigation of Mathematical Abilities in Schoolchildren. In Kilpatrick, J. & Wirszup, I. (Eds.). *Soviet Studies in the Psychology of Learning and Teaching Mathematics*. Chicago, IL: University of Chicago Press. II, 5–57.

McGowen, M. A. 1998. *Cognitive Units, Concept Images, and Cognitive Collages: An Examination of the Process of Knowledge Construction*. Ph.D. Thesis, University of Warwick, U. K.

McGowen, M., DeMarois, P. and Tall, D. 2000. Using the Function Machines as a Cognitive Root. *Proceedings of 22nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tuscon, AZ: 247-254.

McGowen, M.and Tall, D. 1999. Concept maps and schematic diagrams as devices for documenting the growth of mathematical knowledge. In O.Zaslavsky (Ed.). *Proceedings of the 23rd International Group for the Psychology of Mathematics Education*. Haifa, Israel., 3, 281-288.

Shapiro, S. I. (1992). A Psychological Analysis of the Structure of Mathematical Abilities in Grades 9 and 10. In Kilpatrick, J. & Wirszup, I. (Eds.). *Soviet Studies in the Psychology of Learning and Teaching Mathematics*. Chicago, IL: University of Chicago Press.VIII, 97–142.

Tall, D., McGowen, M., & DeMarois, P. 2000. The Function Machine as a Cognitive Root for the Function Concept. *Proceedings of 22nd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Tuscon, AZ: 255-261.

Tall, D., Gray, E., Ali, M. Crowley, L., DeMarois, P., McGowen, M., Pitta, D., Pinto, M., Thomas, M., Yusof, Y. (2000). Symbols and the Bifurcation between Procedural and Conceptual Thinking. *Canadian Journal of Mathematics, Science and Technology Education*, 1 1, 85–109.