$B \subset G$, H spherical

(X, x) simple embedding of G/H

(X, x) contains a unique G-closed orbit Y

$Gx \subset X$ is open dense
stabilizer of x is H

To (X, x) is associated a colored cone.

<u>Players</u>: $N(X) = \mathrm{Hom}(\Lambda(X), \mathbb{Q})$

$\Delta(X)$ = B-inv. prime divisors that are not G-invariant

$\mathcal{D}(X) = \{ D \cap G/H \mid D \in \Delta(X), D \supset Y \}$

<u>Note</u>: If $D \in \Delta(X)$, then $D \cap G/H \neq \emptyset$

ρ_X: $\{\text{discrete valuations}\} \to N(X)$

$\qquad v \longmapsto [\chi_f \mapsto v(f)]$
$\qquad\qquad f \in \mathbb{C}(X)^{(B)}$

$\mathcal{V}(X)$ G-invariant valuations on X

$$\mathcal{C}(X) = \left\langle \varrho_X(\mathcal{D}(X)),\ \varrho_X(v_D) \mid D \underset{\text{prime divisor}}{\overset{G\text{-stable}}{}} \right\rangle$$

$\cap$

$N(G/H)$ The pair $(\mathcal{C}(X), \mathcal{D}(X))$ is called the colored cone of (X, x).

Recall def: Colored cone in $N(G/H)$ is a pair $(\mathcal{C}, \mathcal{D})$

s.t. $\mathcal{C} \subset \mathcal{N}(G/H)$, $\mathcal{D} \subset \Delta(G/H)$

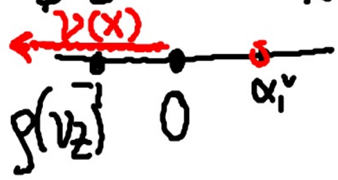
- $\mathcal{C}$ strictly convex polyhedral cone gen. by $\varrho(\mathcal{D})$, $\varrho(\{$finite # of elements in $\mathcal{V}(G/H)\})$
- relint $\mathcal{C} \cap \mathcal{V}(G/H) \neq \emptyset$
- $0 \notin \varrho_{G/H}(\mathcal{D})$.

Theorem: $\left\{\begin{array}{l}(X,x) \text{ simple} \\ \text{spherical embedding}\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}\text{strictly convex} \\ \text{colored cones}\end{array}\right\}$

Example: $G = SL_2$, $H = T$ $X = \mathbb{P}^1 \times \mathbb{P}^1 \circlearrowleft SL(2, \mathbb{C})$ diagonally

- $\Lambda(X) = \mathbb{Z}\alpha_1$ where $\alpha_1 : B \to \mathbb{C}^*$
  $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \mapsto \alpha^{-2}$?

  $\alpha_1 = \chi_f$ where $f = (x-y)^{-1}$
- $N(X) = \mathbb{Q}$ $G/H = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \text{diagonal } Z$
- Z G-invariant, Unique closed G-orbit $\Delta(X) = \{\underbrace{\mathbb{P}^1 \times [1:0]}_{D^+}, \underbrace{[1:0] \times \mathbb{P}^1}_{D^-}\}$
- $\mathcal{D}(X) = \emptyset$ since $D^+ \not\supset Z$, & $D^- \not\supset Z$ N.
- $\rho_X(\nu_Z) = [\alpha_1 \mapsto \nu_Z((x-y)^{-1}) = -1]$



Only possibilities are

$$\left(\{0\}, \emptyset \right) \longleftrightarrow SL_2/T$$

$$\Longrightarrow \left(\underset{0}{\longleftarrow}, \emptyset \right\} \longleftrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

<u>Def</u>: A <u>face</u> a colored cone $(\mathcal{C},\mathcal{D})$ is a colored cone $(\mathcal{C}',\mathcal{D}')$ s.t. $\mathcal{C}'$ is a face of $\mathcal{C}$ and $\mathcal{D}' = \rho^{-1}(\mathcal{C}') \cap \mathcal{D}$

<u>Prop</u>: (X,x) embedding of G/H, Y a closed orbit

[$\rightsquigarrow X_{Y,G} = G \cdot X_{Y,B}$, $X_{Y,B} = X \setminus \bigcup_{\substack{D \text{ B-stable} \\ \text{not containing } Y}} D$ is <u>simple</u> & (X,x) is covered by these]

$\left\{ \begin{array}{l} G\text{orbit of } X \text{ containing } Y \\ \text{in their closure} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{faces of} \\ (\mathcal{C}(X_{Y,G}), \mathcal{D}(X_{Y,G})) \end{array} \right\}$

$Y \subseteq \overline{Z} \subset X$ $\overset{Z}{G\text{orbit}}$ $\longmapsto$ colored cone of $X_{Z,G}$

In fact, we get a colored fan $\mathcal{F}(X)$.

$$\mathcal{F}(X) = \left\{ \begin{array}{l} \text{colored cones associated to } X_{Y,G} \\ \text{where } Y \text{ is } G\text{-orbit in } X \end{array} \right\}$$

<u>Def</u>: Colored fan in $N(G/H)$ is a collection of colored cones s.t.

1) any face of a cone in $\mathcal{F}$ is in $\mathcal{F}$
2) The relative interiors do not intersect

<u>Theorem</u>
$$\left\{ \begin{array}{l} \text{Embeddings } (X,x) \\ \text{of } G/H \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Colored fans} \\ \text{in } N(G/H) \end{array} \right\}$$
$$(X,x) \longrightarrow \mathcal{F}(X).$$

Prop (X, x) is complete

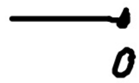
$$\Longleftrightarrow \bigcup_{(\mathcal{C}, \mathcal{D}) \in \mathcal{J}} \mathcal{C} \supseteq \left(\mathcal{V}(G/H) \right)$$

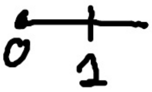
$$\mathcal{V}(G/H) \subset N(G/H)$$

Example: $G = SL_2 \quad H = U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

$f = y$ is B-eigenvector with weight $\omega_1 : \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \mapsto \alpha^{-1}$

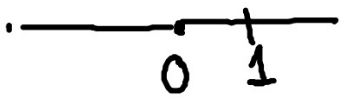
- $\Lambda(G/H) = \mathbb{Z}\omega_1, \quad N = \mathbb{Q}$
- $G/H = \mathbb{C}^2 \setminus \{0,0\}$
- B orbits: $\{y \neq 0\}$ open orbit, 1 closed orbit $\{y = 0\}$ (not G inv.)
- $\mathcal{D}(X) = \{y = 0\}$
- Fact $\varrho(\mathcal{V}(G/U)) = \mathbb{Q} = N(G/U)$
- $\varrho(D) = 1$. $[\omega_1 \mapsto \mathrm{ord}_{\{y=0\}}\, y = 1]$

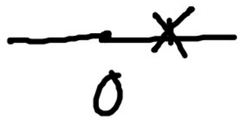
$(\mathcal{C}_1, \mathcal{D}_1) = ($  $, \emptyset) \longleftrightarrow G/H \cup \{\text{line} @ \infty\} = \mathbb{P}^2 \setminus \{pt\}$

$(\mathcal{C}_2, \mathcal{D}_2) = ($  $, \{D\}) \longleftrightarrow G/H \cup \{GC\} = \mathbb{C}^2$

$(\mathcal{C}_3, \mathcal{D}_3) = ($  $, \emptyset) \longleftrightarrow \mathrm{Bl}_{(0,0)} \mathbb{C}^2$

To make a fan, we have 2 options:

$\mathcal{F}_1 = \{(\dot{0}, \emptyset), (\mathcal{C}_1, \mathcal{D}_1), (\mathcal{C}_2, \mathcal{D}_2)\}$  $\longleftrightarrow \mathbb{P}^2$

$\mathcal{F}_0 = \{(\dot{0}, \emptyset), (\mathcal{C}_1, \mathcal{D}_1), (\mathcal{C}_3, \mathcal{D}_3)$  $\longleftrightarrow \mathrm{Bl}_{(0,0)} \mathbb{P}^2$

G/H, G/H' spherical homogneous $H \subset H'$

$\varphi: G/H \to G/H'$

(X, x) (X', x') embeddings of G/H, G/H'

when does φ extend to morphism

$(X, x) \to (X', x')$?

$\varphi : G/H \to G/H'$

induces $\varphi^* : \Lambda(G/H') \hookrightarrow \Lambda(G/H)$ injective

and thus surjective lin. map

$$\varphi_* : N(G/H) \twoheadrightarrow N(G/H')$$

<u>Def</u>: $\mathcal{D}_\varphi := \{ D \in \Delta(G/H) \mid \overline{\varphi(D)} = G/H' \}$

<u>Note</u>: $D \in \mathcal{D}(G/H) \setminus \mathcal{D}_\varphi$, then $\overline{\varphi(D)}$ is a color of G/H'

<u>Theorem</u>: $(X,x)(X',x')$ embeddings of G/H, G/H'. φ extends to homo $X \to X' \iff \mathcal{F}(X)$ dominates $\mathcal{F}(X')$, i.e. for each $(\mathcal{C},\mathcal{D}) \in \mathcal{F}(X)$ $\exists$ colored cone $(\mathcal{C}',\mathcal{D}') \in \mathcal{F}(X')$ s.t. $\varphi_*(\mathcal{C}) \subset \mathcal{C}'$

<u>and</u>: $\varphi_*(\mathcal{D} \setminus \mathcal{D}_\varphi) \subseteq \mathcal{D}'$

D divisor, B-invariant.

D does not intersect G/H.

$G/H \subseteq X$ $\quad X \setminus G/H$ is G-invariant

& codim ≤ 1.