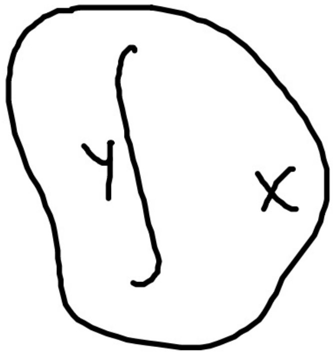
# Pezzini 2.3

"Local Structures"




**Setup:** $X$ - spherical $G$-variety
$Y \subset X$ closed $G$-orbit

**Goal:** Understand a "nice" $G$-invariant neighborhood of $Y$.

Problem: $G$ is just too big.

Soln: Instead consider

$L \subset G$ parabolic subgp

$M \subset X$ subvariety

and consider $L \curvearrowright M$.

Def: A subgroup $P$ is parabolic if $B \subseteq P \subseteq G$ ($B$-borel)

$\Longleftrightarrow$ $P$ parabolic $\Longleftrightarrow$ $G/P$ is a complete variety.

$P < GL_n$ $\qquad\qquad$ $B$

$\updownarrow$ $\qquad\qquad$ $\updownarrow$

partial flag variety $\qquad$ complete flag

$\updownarrow$

show stabilizer is $P$

Example. (Motivation) $G = SL_2$
$X = \mathbb{P}^1 \times \mathbb{P}^1$.
with the diagonal linear action.

- $Z$ = diagonal($\mathbb{P}^1$), is the unique closed $G$-orbit.

  $\longrightarrow$ No $G$-stable neighborhood of $Z$ except for the whole $X$.

Consider B-stable:

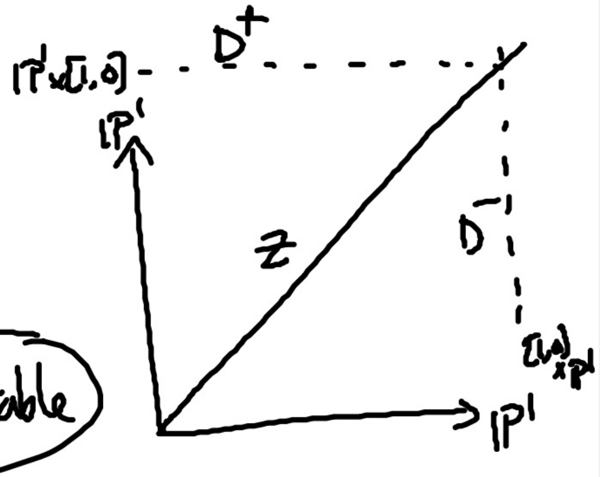
- There are 2 B-stable prime divisors.

$$D^+ = \mathbb{P}^1 \times [1,0] \qquad D^- = [1,0] \times \mathbb{P}^1$$

<u>Define</u>:

$$X_{Z,B} = X \setminus (D^+ \cup D^-)$$

$$= \{[x,1],[y,1]\} \cong \mathbb{A}^2$$




$X_{Z,B}$ is AFFINE B-stable open

Action of $B$ on $X_{2,B} = \left([x,1],[y,1]\right) \cong \mathbb{A}^2$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x,y)$$

<u>Recall</u>: $B = \underset{\substack{\uparrow \\ \text{diagonal}}}{D} \cdot \underset{\substack{\uparrow \\ \text{unipotent}}}{U}$

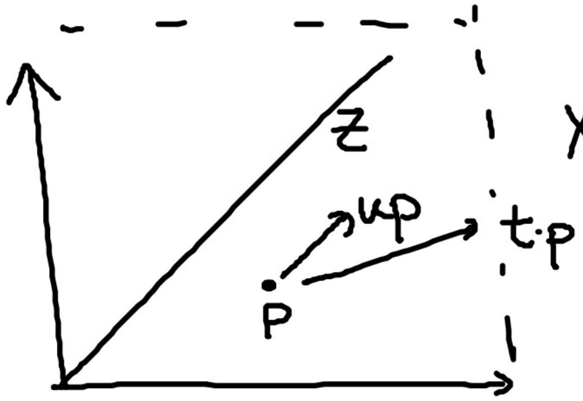
- $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \underset{\substack{\uparrow \\ \mathbb{A}^2}}{(x,y)} \longrightarrow (x+\beta,\, y+\beta)$

  action is translation.

$\begin{bmatrix} \alpha x \\ \alpha^{-1} \end{bmatrix} = \begin{pmatrix} \alpha^2 x \\ 1 \end{pmatrix}$

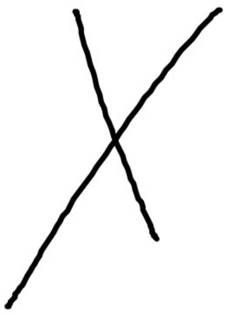
- $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} (x,y) \longrightarrow (\alpha^2 x,\, \alpha^2 y)$

  dilation with weight $\alpha$,




$$X_{Z,B} = X - D^{+} - D^{-}$$

$$\underset{\mathbb{A}^2}{\overset{X_{Z,B}}{\cong}} = (\underbrace{Z \cap X_{Z,B}}_{\text{B-stable}}) \times (\underbrace{\text{line}}_{\text{T-stable.}})$$



$$\{x + y = 0\}$$

<u>Defn</u>: if $Y \subset X$ is a closed $G$-orbit then

$$X_{Y,B} = X \setminus \bigcup D$$

with union over all $B$-stable prime divisors that don't contain $Y$.

<u>Recall</u>: if $P$ is parabolic then

$$P = L \cdot R_u$$

(L is called a Levi subgroup)

maxl red. subgroup. unipotent

(think $B = D \cdot U$)

<u>E.g.</u> $P = \begin{pmatrix} \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ & & \ast & \ast \\ & 0 & \ast & \ast \end{pmatrix}$ then

$$P = \underbrace{\begin{pmatrix} \ast & \ast & & \\ \ast & \ast & & 0 \\ & & \ast & \ast \\ 0 & & \ast & \ast \end{pmatrix}}_{L} \cdot \underbrace{\begin{pmatrix} 1 & & \ast & \ast \\ & 1 & \ast & \ast \\ & & 1 & \\ & & & 1 \end{pmatrix}}_{U}$$

**Thm**: 1. $X_{Y,B}$ is affine, B-stable and equal to $\{x \in X \mid \overline{B \cdot x} \supseteq Y\}$

2. If $Y$ is the unique closed $G$-orbit, then the B-stable prime divisors ~~are~~ not containing $Y$ are all Cartier and gen'd by global section

3. Define $P \supseteq B$ to be the stabilizer of $X_{Y,B}$ and choose a Levi subgroup $L$ of $P$ then $\exists$ affine $L$-spherical $L$-stable closed subvariety $M$ of $X_{Y,B}$ s.t. $(P = L \cdot P^u)$

$$\begin{aligned} P^u \times M &\longrightarrow X_{Y,B} \\ (p,m) &\longrightarrow p \cdot m. \end{aligned}$$

is a $P$ equivariant isomorphism.

Finally $\Lambda(X) = \Lambda(M)$ $(\Rightarrow \mathrm{rk}\, X = \mathrm{rk}\, M)$

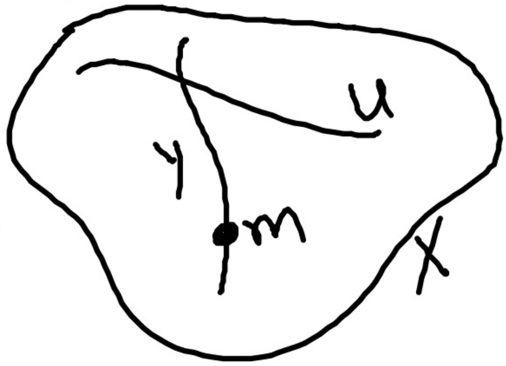
(in our example: $P=B$, $L=T$)

Sketch of Pf :

- Reduce to the case $X = \mathbb{P}(V)$
  ($V$ $G$-module)
- $Y \subset X$ is the unique closed $G$-orbit.
- $U$ $B$-stable hyperplane.

find $m \in Y$ : $\mathrm{stab}(m) = L$

build $M$ explicitly.



**Defn**: A spherical variety is <u>simple</u> if it contains a unique closed $G$-orbit.

**Prop**: Any spherical variety admits a cover by simple $G$-stable $G$-spherical subvarieties.

**Pf**: If $Y \subset X$ is $G$-stable, closed, then

$$G \cdot X_{Y,B} =: X_{Y,G} := \{x \in G \mid \overline{G \cdot x} \supseteq Y\}$$

is open, $G$-stable and has $Y$ as unique closed orbit.

# Dictionary

$X_{Y,B}$ $\longleftrightarrow$ hope of describing $\mathbb{C}[X_{Y,B}]$ via B-weights B-eigenvectors.

open B-stable affine

$X_{Y,B} = P^u \times M$

$X_{Y,G}$ $\longleftrightarrow$ Affine toric Varieties

Simple, G-Spherical open (covered X)

Reference for $G/P$:

Lecture Notes by

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