Invariants & G-stable subsets

$$\begin{array}{ccccc} T & \subset & B & \subset & G \\ & & & & \cap \\ \begin{pmatrix} \ddots & \\ & \ddots \end{pmatrix} & \simeq & \begin{pmatrix} \ddots & \\ 0 & \ddots \end{pmatrix} & \subseteq & GL_n \end{array}$$

X spherical

$$\left(\mathbb{C}(X) \right)^{(B)}$$

To this are associated roots as follows

$T \subset G$

$t \in T \rightsquigarrow \mathrm{Int}(t) : G \to G,\ g \mapsto tgt^{-1}$

G is smooth manifold.

$T_e G = \mathfrak{g}$. So get induced map on tangent space: $\mathrm{Ad}(t) : \mathfrak{g} \to \mathfrak{g}$

$\rightsquigarrow$ get a representation $T \to GL(\mathfrak{g})$.

Example. $G = GL_n$, $\mathfrak{g} = Mat_{n\times n}$

$$Ad(t)\cdot A = tAt^{-1}$$

- $G = SL_n$, $\mathfrak{sl}_n = \{A \in Mat_{n\times n} \mid tr(A)=0\}$
- $G = SL_2 = \left\langle \underset{\overset{\shortparallel}{h}}{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}, \underset{\overset{\shortparallel}{x}}{\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}}, \underset{\overset{\shortparallel}{y}}{\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}} \right\rangle$

$T = \begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}$

$$\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix} h \begin{pmatrix}\alpha^{-1} & 0\\ 0 & \alpha\end{pmatrix} = h$$

$tyt^{-1} = \alpha^{-2}y$

$$\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}\alpha^{-1} & 0\\ 0 & \alpha\end{pmatrix} = \begin{pmatrix}0 & \alpha^2\\ 0 & 0\end{pmatrix} = \alpha^2\cdot x$$

$$\mathrm{Ad}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^{-2} \end{pmatrix}$$

$$\| $$

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$$

$\mathfrak{g}$ decomposes into Eigenspaces w.r.t. the torus action.

$$\mathfrak{g} = \mathfrak{g}^T \oplus \bigoplus_{\chi \in X(T) = M} \mathfrak{g}_\chi ,$$

$$\mathfrak{g}_\chi = \{ v \in \mathfrak{g} \mid t \cdot v = \chi(t) v \}$$

Def The χ for which $\mathfrak{g}_\chi \neq 0$ are called <u>weights</u> of $T \subset G$.

The non-zero weights are called <u>roots</u>.

<u>Example</u>: $G = SL_2(\mathbb{C})$, $T = \mathbb{C}^*$, $X(T) = \mathbb{Z}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha^{-2} \end{pmatrix}$ weights are $-2, 0, 2$ $\alpha_1 = 2$ $\alpha_2 = -2$

roots are $-2, 2$. $\omega_1 = 1$

Let $U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \subseteq B$ unipotent matrices

Note $U \cong \mathbb{A}^N$, $X(U) = \{1\}$

$B = UT$,

$X(B) = X(B/U) = X(T)$

$= \mathbb{Z}^{\dim(T)} = M$

Λ free abelian, its rank is denoted <u>rank of G</u> $= \Lambda$

Theorem 2.4. in Brion

(i) Let V be a simple G-module. Then V^U is a line, B acts via a character $\chi \in \Lambda$ on V. V is uniquely determined, up to G-isom by χ, call $V(\chi)$.

(ii) $\Lambda^+ = \{ \chi \in \Lambda \mid \exists V \in \mathrm{Irr}(G) \text{ s.t. } \chi = \chi(V) \}$ is intersection of Λ with a rational convex cone in $\Lambda_{\mathbb{R}}$. Λ^+ is f.g. submonoid of Λ

<u>Def</u> $\chi(v)$ is called a <u>highest weight</u>.

Elements of Λ^+ are called dominant weights.

A fundamental dominant weight is a generator of Λ^+. $(w_1, w_2, \dots)$

<u>Example</u>: $G = SL_2$, dominant weights are $\mathbb{Z}_{\geq 0}$. The fundamental weight is $w_1 = 1$.

<u>Definition</u>: X spherical with open orbit G/H.

1) $\mathbb{C}(X)^{(B)} = \{ f \in \mathbb{C}(X) \mid b \cdot f = \chi(b) f$ for some $\chi \in \Lambda \}$

2) If $f \in \mathbb{C}(X)^{(B)}$, denote associated character by χ_f.

3) $\Lambda(X) = \{ \chi \in \Lambda \mid \exists f \text{ s.t. } \chi = \chi_f \}$

Λ called the <u>set of associated B-weights</u>.

$\Lambda(X)$ is free abelian group.

The <u>rank of X</u> is defined to be the rank of $\Lambda(X)$.

4) $N(X) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Q})$

this gives the date

<u>Note</u>: if $f, f' \in \mathbb{C}(X)^{(B)}$ s.t. $\chi_f = \chi_{f'}$

$\Rightarrow \frac{f}{f'}$ is constant.

Example: $G = SL_{n+1}$ $\quad X = \mathbb{P}^n = SL_{n+1} \Big/ \begin{pmatrix} * & \begin{matrix} 0 \\ \vdots \\ 0 \\ * \end{matrix} \end{pmatrix}$

$$\underset{\text{(proj)}}{G(k,n)} = SL_{n+1} \Bigg/ \underbrace{\left(\begin{array}{c|c} * & \begin{matrix} 0 \\ \begin{pmatrix} * \end{pmatrix} \end{matrix} \end{array} \right)_{k+1}}_{k+1}$$

These are spherical.

P = parabolic

$$G/P = \coprod_{w \in W} BwP/P$$

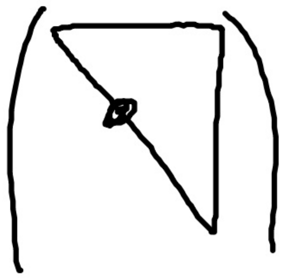
W ← finite group

– one dense orbit, complement must be closed → open

Claim $\operatorname{rank}(G/P) = 0$.

$$\mathbb{C}(X)^{(B)} = \{\text{nonzero constants}\}.$$

$$X(U) = \{1\}$$



$$UP/P = BP/P \text{ open}$$

$$\| \qquad \|$$

$$UTP/P \qquad \Lambda(X) = \{0\}.$$

Example: $G = SL_2 \quad H = T \quad \Lambda(X) = 2\mathbb{Z}$

$$G/H = \{ [x:1], [y:1] \mid x \neq y \}$$
$$\subseteq \mathbb{P}^1 \times \mathbb{P}^1 \setminus \text{diag}$$

G acts diagonally on $\mathbb{P}^1 \times \mathbb{P}^1$.

$$\begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} \cdot \left([x:1], [y:1] \right) = \left([\alpha x + \beta, \alpha^{-1}], [\alpha y + \beta, \alpha^{-1}] \right)$$
$$= \left([\alpha^2 x + \alpha\beta, 1], [\alpha^2 y + \alpha\beta, 1] \right).$$

$$f = (x-y)^n \qquad bf([x:1],[y:1]) = f(\alpha^{-2}x + \alpha^{-1}\beta^{-1}, 1, \alpha^{-2}y + \alpha^{-1}\beta^{-1}, 1)$$
$$= (\alpha^{-2}x - \alpha^{-2}y)^n = \alpha^{-2n}(x-y)$$

Theorem: X spherical G-variety,

then G & B have fin. orbits.

If Y is G-stable, then $Y \subseteq X$ is spherical.