Spherical Varieties

Today: Intro & examples
following Perrini §2.1.

Informal defn:
X variety with a G-action &
a open B-orbit

Formal:
let G be a reductive connected
linear alg group
eg $T^n = (\mathbb{C}^*)^n$ $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$,
$SL_n(\mathbb{C})$.

algebraic group = variety that is
also a group.
& mult & inverse are
morphisms

linear = s.g of GL_n

reductive = (for linear)
every f.d. repn is
semisimple
(direct sum of simples)
(Yes for T^n, SL_n
no for G_a $(K,+)$)

(ref: Brion "Intro to
alg groups")

A Borel s.g of G is
a maximal connected solvable
alg. s.g of G.
eg $G = T^n$ $\underline{B} = T^n$.
$G = SL_n$ $B = \begin{pmatrix} \ast \end{pmatrix}$ upper triang. matrices

<u>Defn</u> Let X be a normal
variety with an (algebraic)
action of a (reductive connected
linear algebraic) group G.
X is a spherical variety
if a Borel s.g $\underline{B}$ has
an open orbit on X.

eg $G = T^n$.
X is spherical if X
is a normal toric variety.
ie X is normal, has a T^n-action,
& a ~~dense~~ open copy of T^n.

eg $G = SL_2$, B = upper triangular matrices.

$X = \mathbb{P}^1$, $G \curvearrowright$ linearly

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} [a:b] = [\alpha a + \beta b : \gamma a + \delta b]$$

The G-action is transitive:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} [0:1] = [\beta : \delta]$$

This means $X = \mathbb{P}^1$ is a homogeneous space (transitive G-action with a base pt).

The stabilizer of $[0:1]$ is

$$\left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} : [\beta : \delta] = [0:1] \right\}$$
$$= \left\{ \begin{pmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{pmatrix} : \alpha \neq 0 \right\}$$

The Borel orbit of $[0:1]$ is

$$\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} [0:1] : \alpha \neq 0 \right\}$$
$$= \{ [\beta : \alpha^{-1}] : \alpha \neq 0 \}$$
$$= \{ [c:1] : c \in \mathbb{C} \}$$
$$= \mathbb{P}^1 \setminus \{[1:0]\}$$

eg $X = \mathbb{P}^1 \times \mathbb{P}^1 \setminus$ diag.

G acts diagonally $g(a,b) = (ga, gb)$.

The action is still transitive:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} ([1:0],[0:1]) = ([\alpha:\gamma],[\beta:\delta])$$

The stabilizer is

$$\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha \neq 0 \right\} \simeq T$$

The Borel gp. orbit of $([0:1],[1:1])$ is

$$\left\{ \begin{pmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{pmatrix} ([0:1],[1:1]) : \alpha \neq 0 \right\}$$
$$\{ [\beta : \alpha^{-1}], [\alpha+\beta : \alpha^{-1}] ; \alpha \neq 0 \}$$
$$= \{ [\alpha\beta : 1], [\alpha(\alpha+\beta) : 1] \}$$
$$= \{ ([c:1],[d:1]) : c \neq d \} \leftarrow \text{open in } \mathbb{P}^1 \times \mathbb{P}^1 \setminus \text{diag}$$

$\alpha = \sqrt{d-c}$ $\quad \beta = \frac{c}{\sqrt{d-c}}$

We can extend the SL_2 action to $\mathbb{P}^1 \times \mathbb{P}^1$, which is still spherical.

eg $X = \mathbb{C}^2 \setminus \{(0,0)\}$

with $G = SL_2$ acting linearly.

$X = G \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. $\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b^{-1} & a \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ if $b \neq 0$

$\begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The stabiliser of $(0,1)$ is

$\left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} : a \in \mathbb{C} \right\} \leftarrow$ unipotent gp.

The B-orbit is

$\{ (x,y) \in X : y \neq 0 \}$

so X is spherical.

We have several options for how to extend the SL_2-action (all spherical)

1) $\overline{X} = \mathbb{C}^2$
2) $\overline{X} = \mathbb{P}^2$. $\quad SL_2 \hookrightarrow PGL(3)$, $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$
3) $\overline{X} = \mathbb{P}^2 \setminus [0:0:1]$
4) $\overline{X} = Bl_{(0,0)}(\mathbb{C}^2)$
$= \{ (x,y), [u:v] : xv = uy \}$
$\subseteq \mathbb{C}^2 \times \mathbb{P}^1$
5) $\overline{X} = Bl_{[0:0:1]}(\mathbb{P}^2)$

<u>Note</u>: In all examples we had two stages:

1) Find a homogeneous space X for G (ie G acts transitively, choice of base pt p), for which X is spherical.

The subgroup
$$H = G_p = \{ g \in G : gp = p \}$$
is called a spherical s.g of G.
$$X = G/H.$$

2). Find compactifications $\overline{X}$ of X on which G-action extends.

2). is somewhat ⟶

(Q: What if $\exists > 1$ conjugacy class of Borels? A) Possibly see a recent student of Brion J. Alg > 2011)

2) is somewhat controlled by the "combinatorics" of G.

eg $G = T^n$

normal toric varieties
$\Updownarrow$
polyhedral fans

Warning:
Not everything from toric geometry carries over.

- Toric varieties are T-stable unions of affine toric varieties.

Not true for spherical varieties
$SL_2 \curvearrowright \mathbb{P}^1$

- Each T-orbit closure is the intersection of codim-ones.

Not true for spherical varieties
$SL_2 \curvearrowright \mathbb{C}^2$