Divisor classes

$Cl(X) = \mathrm{Div}X/\sim$

$\boxed{\text{Prop}}$ (i) $Cl(X)$ is generated by $Z_1, \dots, Z_\ell$ and $D \in \Delta$

(ii) Relations: for $u \in \Lambda$

$$\sum_{i=1}^{\ell} \langle u, \underbrace{v_i}_{\rho(v_{Z_i})} \rangle Z_i + \sum_{D \in \Delta} \langle u, \rho(D) \rangle D$$

<u>proof</u>:

- $X_B^\circ = X \setminus \left(\bigcup_{D \in \Delta} D \cup Z_i \right)$

  $cl(X_B^\circ) = 0$

- $\sum n_i Z_i + \sum n_D D$

  $= div(f) = div(f_w)$

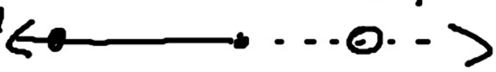
  $= \sum \underbrace{v_{Z_i}(f_w)}_{\langle \rho(v_{Z_i}), w \rangle} \cdot Z_i + \sum_D \underbrace{v_D(f_w)}_{\langle \rho(v_D), w \rangle} D$

Examples: $Sl_2/T = \mathbb{P}^1 \times \mathbb{P}^1 \setminus Z$

$Z = \text{diag.}$

$D_+ = \mathbb{P}^1 \times \infty$

z_1



$\mathbb{P}^1 \times \mathbb{P}^1$

$D_- = \infty \times \mathbb{P}^1$

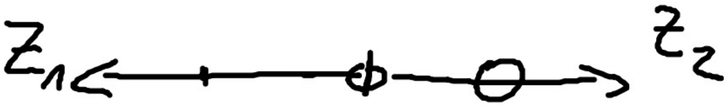
$$-[Z_1] + [D_+] + [D_-] = 0$$

$$Cl(X) = \mathbb{Z}^2$$

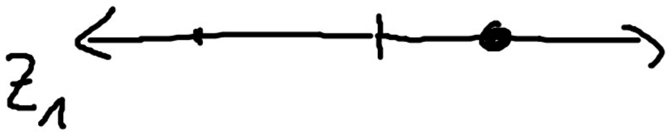
$$\mathrm{div}\left(\frac{x_1}{x_0} - \frac{y_1}{y_0}\right) = Z - D_+ - D_-$$

(ii) $Sl_2 / U = \mathbb{A}^2 \setminus \{0\}$

$\simeq Bl\,\mathbb{P}^2$



$D = \{y = 0\}$



$\simeq \mathbb{P}^2$

1) $-z_1 + z_2 + D = 0$ $\qquad \mathbb{Z}^2$

2) $-z_1 + D = 0$ $\qquad \mathbb{Z}$

X smooth, G (semi-simple) simply connected

$\rightsquigarrow$ every line bundle admits a (unique) G-linearization

$$Cl(X) = Pic(X) = Pic^G(X)$$

in particular we get a (canonical) G-action on $H^0(X, \mathcal{O}(D))$ D a divisor.

Cox ring / total coordinate ring

Assume: (i) X complete

(ii) $Cl(X) \cong \mathbb{Z}^e$

Cox ring

$$R(X) = \bigoplus_{[D] \in Cl(X)} H^0(X, \mathcal{O}(D))$$

$$\tilde{R}(X) = \bigoplus_{[D]} \mathcal{O}(D)$$

$$R(X) = H^0(\tilde{R}(X))$$

<u>Questions</u>: - is $R(X)$ f.g.

- what are the generators and relations

<u>Examples</u>: $\cdot R(\mathbb{P}^n) = k[x_0, \dots, x_n]$

- $R(X \times Y) = R(X) \otimes_k R(Y)$
- $R(X_\Sigma) = k[x_\rho \mid \rho \in \Sigma^{(1)}]$

- $T \curvearrowright \hat{X} \xhookrightarrow[\text{open}]{} \tilde{X} = \operatorname{Spec} R(X)$

$$\begin{tikzcd} \hat{X} = \operatorname{Spec}_X \tilde{R} \arrow[d, bend right] \\ X = \hat{X}/T \end{tikzcd}$$

$\tilde{X} \setminus \hat{X}$ codim > 1

-

Facts for Cox rings of spherical varieties

- $G \curvearrowright \widehat{\mathcal{R}}(X) \qquad G \curvearrowright \hat{X}$
- $G \times T =: \tilde{G} \curvearrowright \hat{X}$

  T having $\mathfrak{X}(T) = Cl(X)$
- $\hat{X}$ is spherical via $\tilde{G}$

$$R(X) = H^0(\hat{X}, \mathcal{O}_{\hat{X}})$$

$$\underset{\text{Knop}}{\Longrightarrow} \quad R(X) \text{ is f.g.}$$

$\rightsquigarrow \quad \tilde{G} \curvearrowright \tilde{X}$ spherical again.

$$H^0(X, \mathcal{O}(E)) \ni s_E$$

$$\underline{\underline{U}} \subset B \subset G$$

$\boxed{\text{Thm}}$ (Brion)

$$R^U(X) = k[s_1, \dots, s_e][s_D \mid$$

$$s_i = s_{E_i} \qquad D \in \Delta]$$

Example: $X = \mathbb{P}^1 \times \mathbb{P}^1$

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

$D^+ = \mathbb{P}^1 \times \infty$

$D^- = \infty \times \mathbb{P}^1$

$Z = \text{diagonal}$

$$R(X) = K[x_0, x_1, y_0, y_1]$$

$$R(X)^U = K[\underbrace{x_0}_{= s_{D_+}}, \underbrace{y_0}_{= s_{D_-}}, \underbrace{x_0 y_1 - x_1 y_0}_{= s_Z}]$$