Spherical Varieties.

Last time: $X_{Y,B} := X \setminus \bigcup D$

B-stable divisor not cont. Y.

Goal: Understand the coord. ring

$\mathbb{C}[X_{Y,B}]$.

§2.4 Discrete Valuations

Def. A map $v : \mathbb{C}(X)^* \longrightarrow \mathbb{Q}$ is a discrete valuation if

- $v(f_1 + f_2) \geq \min(v(f_1), v(f_2))$
- $v(f_1 f_2) = v(f_1) + v(f_2)$
- $v(\mathbb{C}^*) = \{0\}$
- The image of v is discrete.

Recall:

$$\Lambda(X) = \{ \chi_f \mid \underbrace{f \in \mathbb{C}(X)^{(B)}}_{b \cdot f = \chi(b) f} \}$$

weight lattice

$$N(X) = \mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{Q})$$

vector space.

Def: ⓐ D prime divisor on X

denote v_D the associated discrete valuation

$$v_D(f) = \text{"order of vanishing of } f \text{ along } D\text{"}.$$

ⓑ $$\rho_X : \left\{ \begin{array}{c} \text{discrete valuations} \\ \text{on } X \end{array} \right\} \longrightarrow N(X)$$

$$v \longmapsto \rho(v) : \Lambda(X) \to \mathbb{Q}$$

$$\chi_f \longmapsto v(f)$$

$$\rho(v_D) =: \rho(D).$$

Def. A discrete valuation v is <u>G-invariant</u> if $v(g \cdot f) = v(f)$ $\forall g \in G$ $\forall f \in \mathbb{C}(X)$

Ex. D is a G-stable prime divisor then v_D is G-invariant.

$$\mathcal{V}(X) = \left\{ \begin{array}{c} G\text{-invariant discrete} \\ \text{valuations on } X \end{array} \right\}.$$

Def: A <u>color</u> is a prime divisor D which is B-stable but not G-stable.

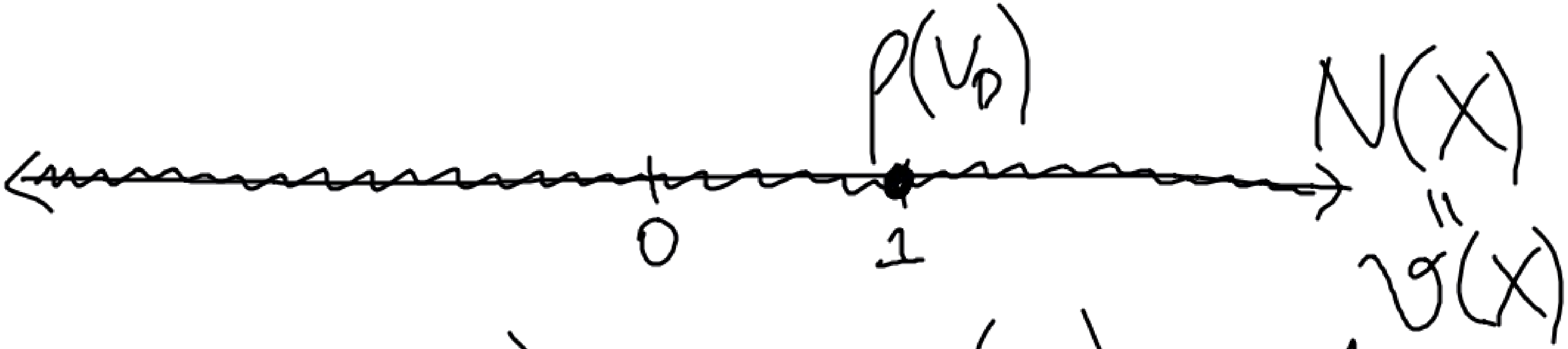
$$\Delta(X) = \{\text{colors on } X\}.$$

- <u>Example</u>: $X = \mathbb{C}^2 - \{(0,0)\}$. $G = SL_2$

$$\Lambda(X) = \mathbb{Z}\omega_1 \to \chi_y$$

Two B-orbits $\{y \neq 0\}$ and $\{y = 0\}$

Only one color $D = \{y = 0\}$



$$\langle \rho(v_D), \underbrace{\omega_1}_{=\chi_y} \rangle = v_D(y) = 1$$

$$\rho(\vartheta(X)) = N(X)$$

If $\eta \in N(X)$, define $v(f) = \min_{n \in \mathbb{N}} \langle \eta, n \cdot \underbrace{\omega_1}_{\chi_y} \rangle$

$\boxed{v(f) \text{ is } G\text{-invariant}}$

f has a part of degree n

Example: $X = \mathbb{P}^1 \times \mathbb{P}^1$ $\quad G = SL_2$ diagonal action

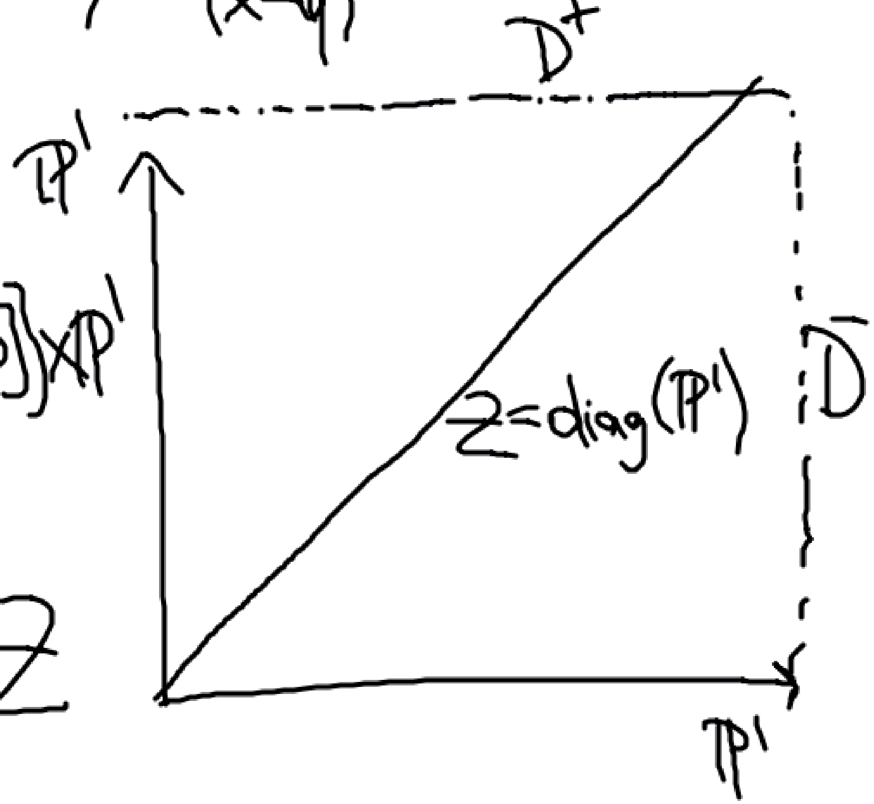
$$\Lambda(X) = \mathbb{Z}\alpha_1 \qquad \alpha_1 = \chi_{(x-y)^{-1}}$$

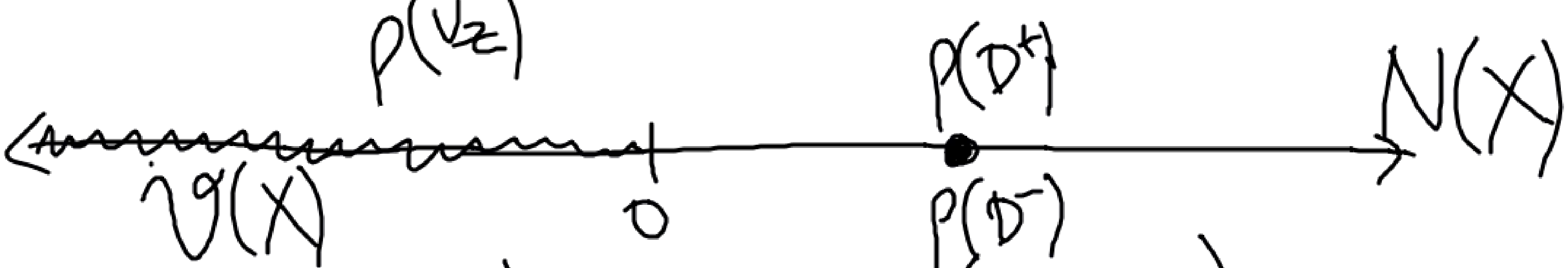
Two B-stable divisors

$D^+ = \mathbb{P}^1 \times \{[1:0]\}$ $\quad D^- = \{[1:0]\} \times \mathbb{P}^1$

(two colors)

One G-stable divisor Z



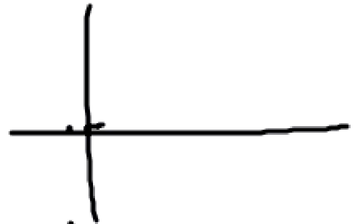


$$\langle \rho(v_Z), \alpha_1 \rangle = v_Z\left((x-y)^{-1}\right) = -1$$

$$\alpha_1 = \chi_{(x-y)^{-1}}$$

Z locally defined by $x-y=0$

$$v_Z \in \mathcal{V}(X)$$



$$\langle \rho(v_{D^+}), \alpha_1 \rangle = v_{D^+}\left((x-y)^{-1}\right) = 1$$

$$\alpha_1 = \chi_{(x-y)^{-1}}$$

$$v_{D^+}\left((x-y)^{-1}\right) = v_{D^-}\left((x-y)^{-1}\right)$$

- **Theorem:** $\rho|_{\mathcal{V}(X)} : \mathcal{V}(X) \to N(X)$

is injective.

"We can recover a G-invariant valuation from how it acts on B-eigenvectors.

Proof sketch:

- Lift v from G/H to G
- If $f \in \mathbb{C}[G]$ let $W = \mathrm{span}\{g \cdot f \mid g \in G\}$

Claim $v(f) = \min_{F \in W^{(B)}} v(F)$ $\boxtimes$

Summary:

- We want to use discrete valuations to understand diff. embeddings of X.
- G-invariant valuations can be thought of as living inside $N(X)$.
- In addition, we need to keep track of the colors.