Structures on Sets of Monomial Ideals

by

Diane Margaret Maclagan

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Committee in charge:

Professor Bernd Sturmfels, Chair
Professor T.Y. Lam
Professor Umesh Vazirani

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The dissertation of Diane Margaret Maclagan is approved:

Chair Date

Date

Date

University of California at Berkeley

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Abstract

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Sets of monomial ideals arise in combinatorics and computational algebra in several different ways. The first such collection considered is the set of all initial ideals of a given ideal in a commutative polynomial ring. Initial ideals, which arise in Gröbner basis theory, enable us to transform questions about the ideal to simpler questions about the related monomial ideal. A universal finiteness theorem for sets of monomial ideals is given, one of whose consequences is a new proof that the set of initial ideals of an ideal is finite. We also study the term orders which give rise to initial ideals, relating them to the Baues problem. Specializing to toric ideals, we then consider the intersection of all initial ideals, called the vertex ideal. Another way a set of monomial ideals arises is as the torus-fixed points of the toric Hilbert scheme. We construct a graph whose vertices are these monomial ideals, which is connected if and only if the scheme is connected. We also give a map from this graph to the Baues graph of triangulations of the associated point configuration.

Professor Bernd Sturmfels
Dissertation Committee Chair

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Notation

Throughout this dissertation we use the following notation.

$\mathbb{N}$		The natural numbers, including zero.
S	$k[x_1, \ldots, x_n]$	The polynomial ring over a field k. The field k will be algebraically closed, but characteristic is arbitrary unless otherwise noted.
x^u	$x_1^{u_1} \ldots x_n^{u_n}$	for $u \in \mathbb{N}^n$.
$u \leq v$	$u, v \in \mathbb{N}^n$	This means $u_i \leq v_i$ for all i.
$[n]$	$\{1, \ldots, n\}$	
A	$[a_1 \ldots a_n]$	A $d \times n$ matrix with integer entries and no zero columns such that $ker(A) \cap \mathbb{N}^n = \{0\}$. The ith column of A is a_i.
$\mathrm{supp}(v)$	$\{i : v_i \neq 0\}$	
Δ		A simplicial complex
$I(\Delta)$		The Stanley-Reisner ideal of Δ.
$\Delta(I)$		The simplicial complex with Stanley-Reisner ideal I.

Chapter 1

Introduction

1.1 Overview

Monomial ideals form one of the interfaces between combinatorics and commutative algebra. A theme of this dissertation is that it can be profitable to study sets of monomial ideals, and that information about an object of interest can often be obtained by studying the structure of these sets.

One of the ways monomial ideals arise in commutative algebra or algebraic geometry is via Gröbner basis theory. One of the fundamental constructions of Gröbner basis theory is to pass from an ideal in a polynomial ring to its *initial ideal*, a monomial ideal which is a flat limit of the original ideal. In the polynomial setting the study of all such initial ideals connects to polyhedral geometry via the *state polytope*, whose vertices correspond to the initial ideals of the ideal. Monomial ideals arising in Gröbner basis theory are the theme of the first part of this dissertation.

Another way this connection between combinatorics and commutative algebra exhibits itself is through the two different ways to define monomial ideals. The standard algebraic definition of a monomial ideal is as an ideal generated by monomials in a polynomial ring in n variables over a field. They can also, however, be viewed as order ideals in the poset $\mathbb{N}^n$. This connection of monomial ideals to poset theory is exploited in Chapter 2, where we show that in the poset of order ideals of $\mathbb{N}^n$ ordered by inclusion all antichains are finite. The algebraic interpretation of this result is that in any infinite collection of monomial ideals there will be a pair of ideals such that one of them is contained inside the other. This fact has many implications for fields as diverse as computational algebra and convex geometry. Much of Chapter 2 is concerned with these implications, including a new proof of a basic theorem in Gröbner basis theory.

In Chapter 3 we consider an aspect of Gröbner basis theory in the exterior algebra over a vector space. The focus of this chapter are the *term orders* in the exterior algebra, which we call *Boolean term orders*. These have also arisen in the realm of comparative probability. Our main result is an interpretation of Boolean term orders as one element extensions of a certain oriented matroid. This places them in the context of the *Baues problem* which is described in the next section.

One aspect of these term orders that is interesting from the Gröbner basis point of view is the existence of *noncoherent term orders*. These are term orders on the exterior algebra that are not induced by term orders on the corresponding polynomial ring. In Chapter 4 we consider such noncoherent term orders in quotients of a polynomial ring by an Artinian monomial ideal. We construct a *noncoherent initial ideal* in this setting. This is an initial ideal of an ideal I with respect to a noncoherent term order, which is not equal to the initial ideal of I with respect to any coherent term order.

In Chapter 5 we consider all initial ideals of a given lattice ideal simultaneously, by studying their common intersection. This new monomial ideal, called the *vertex ideal*, gives information about geometric properties of the lattice. We give several algorithms to compute it, and study its associated primes. This leads to a counterexample to a conjecture about the complexity of codimension three toric ideals. This is joint work with Serkan Hoşten.

In the second part of this dissertation we study particular sets of monomial ideals, the monomial A-graded ideals. This is joint work with Rekha Thomas. Monomial A-graded ideals are the ideals corresponding to torus-fixed points on the *toric Hilbert scheme*. In Chapter 6 we describe a graph whose vertices are these ideals. We show that this graph is connected if and only if the corresponding toric Hilbert scheme is connected, and give a map from this graph to the *Baues graph*. We also exhibit toric Hilbert schemes of arbitrarily high dimension. One open problem in the study of toric Hilbert schemes is the question of whether they are always connected. In Chapter 7 we present some approaches and partial results to this problem.

1.2 Background

This section contains a brief exposition of some of the concepts used in the rest of the dissertation. These ideas can almost all be found in the book *Gröbner Bases and Convex Polytopes* [43].

1.2.1 Gröbner Bases

Probably the most painless way to learn about Gröbner bases is to read the first two chapters of *Ideals, Varieties and Algorithms* [7]. We present the key ideas here.

One of the fundamental notions of Gröbner bases is that of a *term order*. This is a total order on the monomials of $S = k[x_1, \ldots, x_n]$ satisfying two properties:

- $1 \prec x^u$ for all $u \neq 0$.
- If $x^a \prec x^b$ then $x^a x^c \prec x^b x^c$.

A standard term order is the lexicographic term order, defined by setting $x^u \prec_{lex} x^v$ if $j = \min\{i : (v-u)_i \neq 0\}$ has the property that $v_j > u_j$. Term orders in rings other than the polynomial ring are considered in Chapters 3 and 4. A term order lets us define the *initial term* of a polynomial $f = \sum_i^l c_i x^{u_i}$ to be the maximum $x^{u_i} = in_\prec(f)$ such that $c_i \neq 0$, where the maximum is taken with respect to the term order $\prec$. For example, if the polynomial ring is $k[a,b,c,d]$, and $f = bc - ad$, then the leading term of f with respect to the lexicographic order with $a \succ b \succ c \succ d$ is bc. The *initial ideal* of an ideal I with respect to the term order $\prec$ is now the monomial ideal

$$in_\prec(I) = \langle in_\prec(f) : f \in I \rangle.$$

A set $G \subseteq I$ is a *Gröbner basis* for I if the ideal generated by $in_\prec(G) = \{in_\prec(f) : f \in G\}$ is equal to $in_\prec(I)$. For example, if $I = \langle ab - c^2, b^3 - c^3 \rangle$, then a lexicographic Gröbner basis is $\{ab - c^2, b^3 - c^3, ac^3 - b^2c^2\}$. The standard algorithm to compute Gröbner bases is the *Buchberger algorithm*. For details see [7, Chapter 2].

One way to define a term order on a polynomial ring is to use a *weight vector* $\omega \in \mathbb{N}^n$. We set $x^u \prec x^v$ if either $\omega \cdot u < \omega \cdot v$, or $\omega \cdot u = \omega \cdot v$ and $x^u \prec_{lex} x^v$. Any other fixed term order can take the place of the lexicographic term order as the "tie-breaker" term order. Alternatively we can not use a tie-breaker term order, and instead take the *initial form* with respect to the partial order on monomials given by ω (sum of the terms of highest degree) instead of the initial term. If ω is a sufficiently generic vector this ideal $in_\omega(I)$ will be a monomial ideal, and will be equal to the initial ideal with respect to the term order determined by ω.

There is a polytope, called the *state polytope* of I, whose vertices are in bijection with the (finitely many) different initial ideals of I. We give a new proof of the fact that I has only finitely many different initial ideals in Chapter 2. If ω is in the interior of the normal cone at a vertex v of the state polytope, then the initial ideal corresponding to v is $in_\omega(I)$.

1.2.2 Toric Ideals

Given a $d \times n$ integer matrix A, a *circuit* of A is an element of the kernel of A with minimal support (set of non-zero entries). We also use the word circuit to denote binomials $x^u - x^v$ with $Au = Av$ with minimal support. If $x^u - x^v$ is a circuit, then $u - v$ is a circuit, and vice versa. The circuits are elements of a larger, finite, subset of the kernel of A, called the *Graver basis* of A. This is defined as follows.

Definition 1.2.1. The *Graver basis* Gr_A of a matrix A is the set of binomials $x^u - x^v$ with $Au = Av$ such that there do not exist $u', v' \in \mathbb{N}^n$ with $Au' = Av'$ and $u' < u$, $v' < v$. Here we set $u < v$ if for each i we have $u_i < v_i$. We also denote by Gr_A the set of vectors $\{u - v : x^u - x^v \in Gr_A\}$. In this formulation Gr_A is the union of Hilbert bases for the cones given by intersecting $ker(A)$ with each orthant of $\mathbb{Z}^n$. This shows that Gr_A is a finite set.

The toric ideal, I_A, associated to a $d \times n$ integer matrix A is the kernel of the map $\pi : S = k[x_1, \ldots, x_n] \to k[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ given by $\pi : x_i \mapsto t^{a_i}$. Generators for this ideal are:

$$I_A = \langle x^u - x^v : Au = Av \rangle.$$

For example if A is the 2 by 4 matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix},$$

then the toric ideal is $I_A = \langle bc - ad, c^2 - bd, b^2 - ac \rangle$. A finite (though usually nonminimal) generating set for I_A is given by the Graver basis of A.

1.2.3 Simplicial Complexes

An abstract simplicial complex is a collection Δ of subsets of $[n] = \{1, \ldots, n\}$ which is closed under taking subsets, so if $\sigma \in \Delta$ and $\tau \subseteq \sigma$ then $\tau \in \Delta$. A geometric simplicial complex $|\Delta|$ is a union of geometric simplices such that the intersection of any two is a face of each. Given a geometric simplicial complex, the associated abstract simplicial complex has maximal simplices the sets of vertices of each simplex. Given an abstract simplicial complex Δ we can construct a geometric simplicial complex in $\mathbb{R}^n$ which realizes it by taking the simplices $\mathrm{conv}(e_i : i \in \sigma)$ for each $\sigma \in \Delta$.

An algebraic concept associated to a simplicial complex is its *Stanley-Reisner ideal*. By a squarefree monomial ideal we mean an ideal generated by squarefree monomials.

Definition 1.2.2. The *Stanley-Reisner ideal* of an abstract simplicial complex Δ is the squarefree monomial ideal

$$I(\Delta) = \langle \prod_{i \in \sigma} x_i : \sigma \notin \Delta \rangle.$$

$I(\Delta)$ is generated by the minimal non-faces of Δ. Given a squarefree monomial ideal I, we can form the simplicial complex

$$\Delta(I) = \{\sigma \subseteq [n] : \prod_{i \in \sigma} x_i \notin I\}.$$

Clearly this is a bijection, with $I(\Delta(I)) = I$ and $\Delta(I(\Delta)) = \Delta$. Examples of this correspondence are shown in Figure 1.1.

There is an entire theory relating algebraic invariants of the Stanley-Reisner ideal to combinatorial invariants of the simplicial complex, but we will use only the definition. Exposition of some of this theory can be found in [40].

1.2.4 The Baues Problem

A good reference for all forms of the Baues problem is [36]. A triangulation of a point configuration $\mathcal{A} = \{a_1, \ldots, a_n\} \subseteq \mathbb{R}^d$ is a geometric simplicial complex covering $\mathrm{conv}(a_1, \ldots, a_n)$ with the vertices of each simplex being a subset of $\mathcal{A}$. Each simplex σ is indexed by the set $\{i : a_i \text{ is a vertex of } \sigma\}$.

A basic operation on triangulations of a point configuration is the *bistellar flip*. The two basic types of non-degenerate bistellar flips in the plane are shown in Figure 1.2.

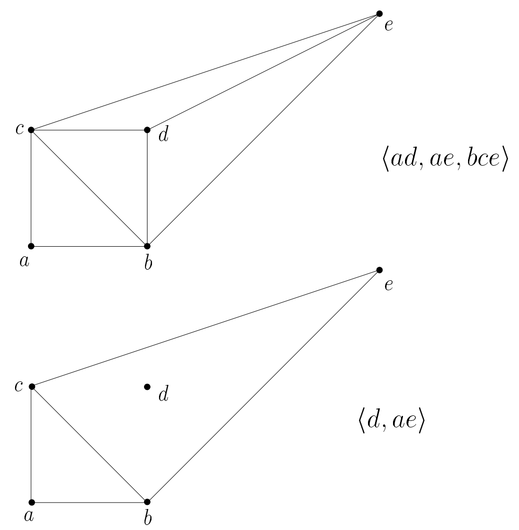

Figure 1.1: The Stanley-Reisner correspondence

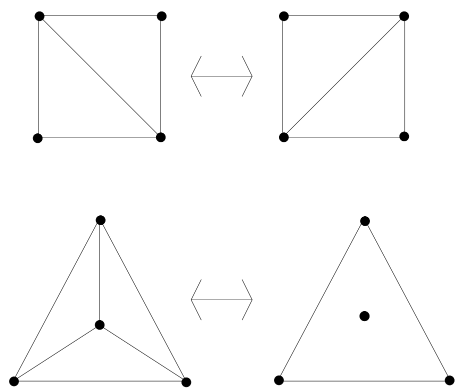

Figure 1.2: Bistellar flips for triangulations of points in the plane

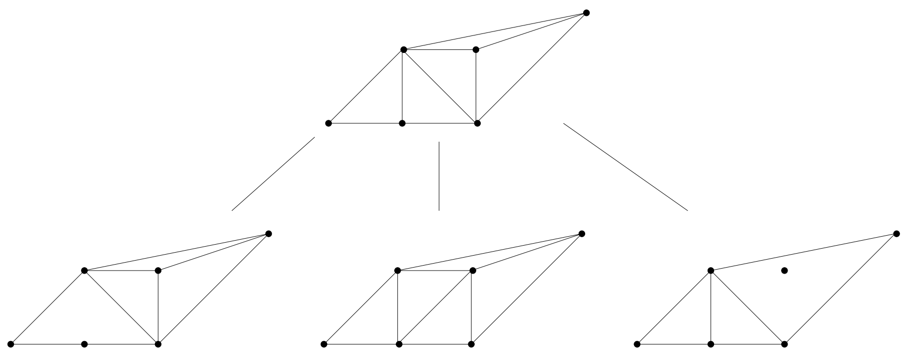

Figure 1.3: Triangulations differing by bistellar flips

Intuitively, a bistellar flip should be thought of as gluing in a higher dimensional simplex, and then turning that simplex over and viewing it from the other side. This can be seen most clearly in the second example in Figure 1.2, which can be viewed as the top and bottom of a tetrahedron. The first example can also be thought of as two opposite views of a tetrahedron.

Examples of bistellar flips are shown in Figure 1.3.

We can form a graph, called the *Baues graph*, with vertices the set of all triangulations of a point configuration, and an edge connecting two triangulations when they differ by a bistellar flip. Figure 1.3 is a subgraph of the Baues graph for a particular collection of six points in the plane. An obvious question to ask is whether the Baues graph is connected. Santos recently answered this question negatively [38] by constructing a configuration of 324 points in $\mathbb{R}^6$ which has a disconnected Baues graph.

We can also form a poset of *subdivisions* of the point configuration, ordered by refinement. A standard construction in algebraic combinatorics is to form the *order complex* of a poset, which is a simplicial complex whose vertices correspond to elements of the poset, and whose simplices are chains in the poset. One form of the *Baues problem* asks if the order complex of the subdivision poset has the homotopy type of a sphere. There are many other formulations of the Baues problem, including a version for zonotopal tilings and the extension space conjecture for oriented matroids [47]. For details see [36].

We close with a formal definition of a bistellar flip, which we will use in Chapter 6. A bistellar flip interchanges the two different triangulations of a *circuit* (minimal affine dependence) of $\mathcal{A}$. Let t be a circuit of the configuration $\mathcal{A}$, and $T = \{i : t_i \neq 0\}$ be its support. We denote by T^+ the set $\{i : t_i > 0\}$ and by T^- the set $\{i : t_i < 0\}$. There are exactly two triangulations of $C = \mathrm{conv}(a_i : i \in T)$. The first, C^+, has $|T^+|$ simplices, which are the simplices indexed by the sets $\{T \setminus \{i\} : i \in T^+\}$. The second, C^-, has $|T^-|$ simplices, which are the sets in $\{T \setminus \{i\} : i \in T^-\}$. The unique minimal non-face of C^+ (respectively C^-) is T^+ (respectively T^-). If C is d-dimensional, and one of C^+ and C^- is a subcomplex of the triangulation Δ, then a bistellar flip over the circuit t involves replacing the subcomplex C^+ by C^- or vice versa.

If C is lower dimensional, we impose an additional condition for t to be flippable. By the *link* of a simplex σ in a simplicial complex Δ we mean the collection of simplices $\{\tau : \tau \cap \sigma = \emptyset, \tau \cup \sigma \in \Delta\}$. We say t is flippable if C^+ (or C^-) is a subcomplex of Δ, *and* the link in Δ of every maximal simplex of C^+ (respectively C^-) is the same subcomplex L. This second condition is trivially satisfied if C is d-dimensional, as the link of every maximal simplex is the empty set. A bistellar flip over the circuit t from C^+ to C^- then involves replacing the simplices $\{l \cup \sigma : \sigma \in C^+, l \in L\}$ by the simplices $\{l \cup \tau : \tau \in C^-, l \in L\}$.

Part I

Monomial ideals in Gröbner Basis Theory

Chapter 2

Antichains of Monomial Ideals are Finite

2.1 Introduction

The main result of this chapter is the following theorem.

Theorem 2.1.1. *Let $\mathcal{I}$ be an infinite collection of monomial ideals in a polynomial ring over an arbitrary field. Then there are two ideals $I, J \in \mathcal{I}$ with $I \subseteq J$.*

If the monomial ideals were all principal, this would be Dickson's Lemma (see Chapter 2, Section 4 of [7]), or a special case of the Hilbert Basis theorem. The word monomial is necessary, as can be seen from the collection of ideals $\{\langle x - a\rangle : a \in k\}$ where $S = k[x]$ and k is a infinite field.

Although the statement of Theorem 2.1.1 may appear to be purely algebraic, monomial ideals are highly combinatorial objects. In particular, the above theorem can be restated as follows:

Theorem 2.1.2. *Let L be the poset of order ideals of the poset $\mathbb{N}^n$, ordered by containment. Then L contains no infinite antichains.*

We note that although there are no infinite antichains there exist arbitrarily large antichains. For $n = 2$ an antichain of size k is given by the set of principal order ideals generated by the elements $\{(i, j) : i + j = k - 1\}$. A special case of the main theorem pertains to Young's lattice, which consists of the set of all partitions ordered by containment of Ferrers diagrams. Noting that a partition can be considered to be a finite order ideal in $\mathbb{N}^2$, we consider the generalized Young's lattice of finite order ideals in $\mathbb{N}^n$ ordered by inclusion.

Theorem 2.1.3. *All antichains in the generalized Young's lattice are finite.*

In the next section we give some corollaries of Theorem 2.1.1. Some of the corollaries have appeared in the literature before, but Theorem 2.1.1 allows us to simplify the original proofs, and provides a common framework for finiteness results involving monomial ideals. In Section 3 we give an application to SAGBI bases which was the motivating example for this paper. In Section 4 we outline an example which shows that one natural generalization to more abstract posets is false, and lastly in Section 5 we give a proof of the theorem.

2.2 Corollaries

In this section we give several corollaries of Theorem 2.1.1. The first corollary is a new proof of a basic result in computational algebra. The following theorem appears in [5] and [29], and is well known.

Corollary 2.2.1. *For a given ideal $I \subseteq S$ there are only finitely many distinct initial ideals $in_{\prec}(I)$.*

Proof. The monomials of S outside $in_{\prec}(I)$ form a k-basis for S/I. If there were infinitely many initial ideals then Theorem 2.1.1 would give a proper inclusion of k-bases. □

Given an $\mathbb{N}^d$ grading on S, such that $\deg(x_i) \neq 0$ for all i, we can define the Hilbert series of a homogeneous ideal I by

$$H_{S/I}(t) = \sum_{b \in \mathbb{N}^d} (\dim_k (S/I)_b) t^b.$$

Corollary 2.2.2. *With respect to a fixed grading there are only finitely many monomial ideals with a given Hilbert series.*

□

Theorem 2.1.2 is also true when $\mathbb{N}^n$ is replaced by a finitely generated submonoid (such as the lattice points inside a rational cone).

Corollary 2.2.3. *Let M be a finitely generated submonoid of $\mathbb{N}^n$. Let $R = k[M] = k[t^{a_1}, \dots, t^{a_l}]$ be its monoid algebra, where the a_i are the generators of M. A monomial ideal in R is an ideal generated by elements of the form $t^b \in R$ for some $b \in \mathbb{N}^n$. Then in any infinite collection $\mathcal{I}$ of monomial ideals in R there are two, $I, J \in \mathcal{I}$, such that $I \subseteq J$.*

Proof. Consider the map $\phi : k[x_1, \dots, x_l] \to R$ given by $\phi : x_i \mapsto t^{a_i}$. For a monomial ideal $I \subseteq R$, we define $I_\phi = \langle x^a : \phi(x^a) \in I \rangle$. Then $I_\phi \subseteq J_\phi \Rightarrow I \subseteq J$, so the result follows from applying Theorem 2.1.1 to the set $\mathcal{I}_\phi = \{I_\phi : I \in \mathcal{I}\}$. □

A similar corollary relates to A-graded algebras, where A is a $d \times n$ matrix with entries in $\mathbb{N}$. An A-graded algebra is a k-algebra R generated by $x_1, x_2, \dots, x_n$ with an $\mathbb{N}^d$ grading (given by $\deg x_i = a_i$, where a_i is the ith column of A) such that $\dim_k R_b = 1$ whenever $b \in \mathbb{N}A$ and equals 0 otherwise. Such algebras are studied in Part II.

Corollary 2.2.4. *Let R be an A-graded algebra. Let $\mathcal{I}$ be an infinite collection of ideals of R which are homogeneous with respect to the A-grading. Then there are two ideals, $I, J \in \mathcal{I}$ such that $I \subseteq J$.*

Proof. R is isomorphic to S/I for some binomial ideal I. Any element of S/I which is homogeneous with respect to the $\mathbb{N}^d$ grading can be written as $m + I$ where m is some monomial in S, so homogeneous ideals of R lift to monomial ideals in S. Containment in S implies containment in R, so the result follows. □

A trivial example of an A-graded algebra is $k[x_1, \dots, x_n]$ with A the $n \times n$ identity matrix. Then Corollary 2.2.4 reduces to Theorem 2.1.1.

2.3 Application to SAGBI bases

Let $T = R[c_1 x^{a_1}, \dots, c_n x^{a_n}]$ be a monomial subalgebra of $R[x_1, \dots, x_d]$, where R is a Principal Ideal Domain. A strong SAGBI (Subalgebra Analogue to Gröbner Bases for Ideals) basis for T is a collection $\{k_1 x^{b_1}, \dots k_m x^{b_m}\}$ such that any element $cx^l \in T$ can be written as $cx^l = r \prod_{i=1}^{m} (k_i x^{b_i})^{\phi_i}$ for some $\phi \in \mathbb{N}^m$ and $r \in R$.

Definition 2.3.1. Given a matrix $A \in \mathbb{N}^{d \times n}$, we define a map $\pi : \mathbb{N}^n \to \mathbb{N}^d$ by $\pi : y \mapsto Ay$. Let $\mathbb{N}A \subseteq \mathbb{N}^d$ be the image of π. For $b \in \mathbb{N}A$ let $P_b = conv(\pi^{-1}(b))$. Since $\pi^{-1}(b)$ is a finite set, this is a convex polytope. We call P_b the fiber of A over b. A fiber over b is *atomic* if there do not exist $b_1, b_2 \in \mathbb{N}^d$ with $b_1 + b_2 = b$ such that $P_b = P_{b_1} + P_{b_2}$, where the addition is Minkowski sum.

Atomic fibers were defined by Adams et al. in [1], where they proved that there are only a finite number of atomic fibers for a given matrix A. They used this result to construct a finite strong SAGBI basis as follows:

Theorem 2.3.2 (Adams et. al. [1]). *Let $T = R[c_1 x^{a_1}, \dots, c_n x^{a_n}]$. Let $A = [a_1, \dots, a_n]$ be the $d \times n$ matrix with columns the a_i. Then $\{k_b x^b : P_b$ is an atomic fiber of $A\}$ is a strong SAGBI basis for T, where $k_b = \gcd(\{c^u = c_1^{u_1} \dots c_l^{u_l} : u = (u_1, \dots, u_n) \in \pi^{-1}(b)\})$.*

The proof of the finiteness result in [1] was constructive but complicated, using convex geometry techniques. Theorem 2.1.1 gives a much simpler, though non constructive, proof of this result.

Corollary 2.3.3. *For a given matrix $A \in \mathbb{N}^{d\times n}$, there are only a finite number of atomic fibers.*

Proof. For $b \in \mathbb{N}A$, let $I_b = \langle x^u : Au = b$ and u is a vertex of $P_b\rangle$. Then the fiber over b is atomic if and only if I_b is not contained in any $I_{b'}$ for $b \neq b'$. If there were an infinite number of atomic fibers, then $\{I_b : P_b$ atomic$\}$ would be an infinite antichain of monomial ideals, contradicting Theorem 2.1.1. □

Corollary 2.3.3 can be generalized as follows:

Definition 2.3.4. Let M be a monomial ideal of S, and $A \in \mathbb{N}^{d\times n}$ a matrix. Then the (M, A) fiber over $b \in \mathbb{N}A$ is the set $\{u : Au = b$ and $x^u \notin M\}$. A (M, A) fiber over b is atomic if there do not exist $b_1, b_2 \in \mathbb{N}A$ with $b_1 + b_2 = b$ such that for all u in the (M, A) fiber over b there are u_1, u_2 in the (M, A) fibers over b_1, b_2 respectively such that $u = u_1 + u_2$.

To see that this definition is a generalization of an earlier one, we first need another definition.

Definition 2.3.5. Given a matrix $A \in \mathbb{N}^{d\times n}$, we define its *vertex ideal*, V_A by

$$V_A = \bigcap_{\prec} in_{\prec}(I_A)$$

where the intersection is over all term orders $\prec$, and I_A is the toric ideal corresponding to A.

The vertex ideal is the object of study in Chapter 5. Note that this intersection is finite by Corollary 2.2.1. Since the standard monomial of A-degree b of an initial ideal of a toric ideal corresponds to a vertex of P_b, and each vertex of P_b is standard for some initial ideal, the set of standard monomials of V_A is exactly $\{x^u : u \text{ is a vertex of } P_{Au}\}$. Thus Definition 2.3.1 is Definition 2.3.5 with $M = V_A$.

Corollary 2.3.6. *There are only finitely many atomic (M, A) fibers for given M and A.*

The proof is the same as for Corollary 2.3.3. Of particular interest is the case $M = (0)$. In that case, being atomic corresponds to the nonexistence of b_1, b_2 such that each lattice point in $\pi^{-1}(b)$ is a sum of lattice points in $\pi^{-1}(b_1)$ and $\pi^{-1}(b_2)$, as opposed to the original definition, where only the vertices need be sums of lattice points in the two smaller fibers. This is a strictly stronger requirement. The following example shows that a fiber can be atomic with respect to this stronger definition without being atomic in the original sense.

Example 2.3.7. Let A be the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 & 0 \\ 5 & 0 & 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Let $b_1 = (1, 3, 5, 2)^T$, and $b_2 = (5, 10, 10, 6)^T$. We have

$$\begin{aligned} \pi^{-1}(b_1) = \{ \quad & (1, 0, 0, 3, 0, 2)^T, \\ & (0, 1, 0, 0, 5, 0)^T, \\ & (0, 0, 1, 1, 3, 1)^T\} \end{aligned}$$

and

$$\begin{aligned} \pi^{-1}(b_2) = \{ \quad & (0, 0, 5, 0, 0, 1)^T, \\ & (1, 2, 2, 0, 1, 0)^T, \\ & (2, 3, 0, 1, 0, 0)^T\} \end{aligned}$$

Now $P_{b_1+b_2} = P_{b_1} + P_{b_2}$, so b_1+b_2 is not atomic in the first sense. However $(1,1,4,2,2,2)^T \in \pi^{-1}(b_1+b_2)$, but cannot be written as the sum of lattice points in P_{b_1} and P_{b_2}. This example is based on an example of Oda [30] for lattice polytopes.

2.4 Poset Formulation

From the second formulation of the theorem, it is natural to suspect that this is in fact a general theorem about posets. Two properties of the poset $\mathbb{N}^n$ which lend themselves to finiteness results are that $\mathbb{N}^n$ has no infinite antichains, and satisfies the descending chain condition. Such posets are known as well-quasi-ordered. The following example consists of a poset which has no infinite antichains or infinite descending chains such that the poset of dual order ideals under containment contains an infinite antichain. This example appears in [9], but was discovered independently by George Bergman, from whom I learned it.

Example 2.4.1. *([9], G. Bergman)* Let X be the set $\{(i,j) : i,j \in \mathbb{N}, i < j\}$. Set $(i,j) \prec (i',j')$ if and only if $j < j'$ and either $i = i'$ or $j < i'$. It is straightforward to check that X is a partially ordered set.

Note that any chain descending from (i,j) can have at most $j-1$ members less than (i,j), so there are no infinite descending chains of elements of X. To see that all antichains in X are finite, suppose Y is an antichain in X, and let j_0 be the smallest j such that $(i,j) \in Y$, occurring in the pair $(i_0,j_0) \in Y$. Then $(i,j) \in Y$ implies $i \leq j_0$, as otherwise $j_0 < j$, and then $(i_0,j_0) \prec (i,j)$. If there are two pairs $(i,j),(i,j') \in Y$, with $j < j'$ then $(i,j) \prec (i,j')$, so there is only one pair of the form $(i,j) \in Y$ for each value of i. But this means there are at most j_0+1 elements in Y, so all antichains in X are finite.

Because there are no infinite descending chains or infinite antichains each dual order ideal in X can be represented by its finite antichain of minimal elements. One dual order ideal is contained in another exactly when each element of the finite antichain of minimal elements of the first dual order ideal is greater than some element of the finite antichain of minimal elements of the second.

For fixed $l > 0$, let $S_l = \{(k,l) : k < l\} \subseteq X$. Then S_l is the finite antichain of minimal elements of a dual order ideal of X. Suppose the dual order ideal determined by S_{l_2} is contained in the one determined by S_{l_1}. From above, we must have $l_1 < l_2$. But then there is no element of S_{l_1} less than $(l_1,l_2) \in S_{l_2}$, a contradiction. So the S_l form an infinite antichain of dual order ideals of X.

Theorem 2.1.2 has, however, recently been generalized in the following way:

Theorem 2.4.2. *(Farley, Schmidt) [13] Let P and Q be two posets with no infinite antichains that satisfy the descending chain condition. If the posets of dual order ideals of P and of Q, ordered by inclusion, have no infinite antichains, then the same is true for the poset of dual order ideals of $P \times Q$.*

2.5 Proof of the Main Theorem

In this section we give the proof of Theorem 2.1.1.

We first prove Theorem 2.1.3. The generalized Young's lattice is isomorphic to the poset of zero-dimensional monomial ideals under inclusion, via the map taking an order ideal to its complement, so we prove the theorem in that setting.

Lemma 2.5.1. *Let $\mathcal{I}$ be an infinite collection of zero-dimensional monomial ideals (primary to the irrelevant maximal ideal). Then there are two ideals, $I, J \in \mathcal{I}$ such that $I \subseteq J$.*

Proof. Suppose $\mathcal{I}$ consists of an infinite number of zero-dimensional monomial ideals, which are noncomparable with respect to inclusion. Choose $I_1 \in \mathcal{I}$. Since $I \not\subseteq I_1$ for $I \in \mathcal{I} \setminus \{I_1\}$, each $I \in \mathcal{I} \setminus \{I_1\}$ contains some of the finite number of standard monomials of I_1. There are thus an infinite number of ideals in $\mathcal{I}$ which contain the same set of standard monomials of I_1. Call this collection $\mathcal{I}_1$. Let J_1 be the intersection of the ideals in $\mathcal{I}_1$. We will now build a strictly ascending chain of monomial ideals. Suppose $\mathcal{I}_k$ and J_k have been chosen. Choose an ideal $I_{k+1} \in \mathcal{I}_k$. We can again find an infinite collection of ideals in $\mathcal{I}_k$ which have the same non-trivial intersection with the standard monomials of I_{k+1}. Let $\mathcal{I}_{k+1}$ be this collection, and let J_{k+1} be the intersection of the ideals in $\mathcal{I}_{k+1}$. We have $J_{k+1} \supsetneq J_k$, since J_{k+1} contains some standard monomials of I_{k+1}, so in this fashion we get an infinite ascending chain of monomial ideals in S, which is impossible. ☐

Corollary 2.5.2. *Let $\mathcal{I}$ be an infinite collection of zero-dimensional monomial ideals. Then there is an infinite chain $I_1 \supsetneq I_2 \supsetneq \ldots$ of ideals in $\mathcal{I}$.*

Proof. Since S is Noetherian, $\mathcal{I}$ contains ideals which are maximal with respect to inclusion of ideals in $\mathcal{I}$. There are only finitely many such maximal ideals by Lemma 2.5.1, so set I_1 to be a maximal ideal in $\mathcal{I}$ containing an infinite number of ideals of $\mathcal{I}$, and repeat, setting $\mathcal{I} = \{I \in \mathcal{I} : I \subsetneq I_1\}$. ☐

Proof of Theorem 2.1.1. Every associated prime of a monomial ideal is a monomial prime, of which there are only a finite number. We can thus restrict to an infinite collection of $\mathcal{I}$ all of which have the same set of associated primes, which we will also call $\mathcal{I}$. Now for each ideal in this set we find an irredundant primary decomposition, writing the ideal as the intersection of monomial ideals primary to an associated prime in such a way that each associated prime is used only once. Let I_τ be the primary component of I primary to the monomial prime $P_\tau = \langle x_i : i \notin \tau \rangle$, where $\tau \subseteq [n]$. For a fixed τ either $\{I_\tau : I \in \mathcal{I}\}$ is finite, so there is an infinite number of $I \in \mathcal{I}$ with the same I_τ, or we can apply Corollary 2.5.2 to the polynomial ring $k[x_i : i \notin \tau]$. In either case we get an infinite collection $\mathcal{I}_\tau = \{I_k : k \geq 1\}$ of ideals in $\mathcal{I}$ such that $I_{1\tau} \supseteq I_{2\tau} \supseteq \ldots$, where the inclusions need not be proper. Since there are only a finite number of associated primes of ideals in $\mathcal{I}$, by appropriate restrictions we can find a sequence $\{I_k : k \geq 1\}$ such that $I_{1\tau} \supseteq I_{2\tau} \supseteq \ldots$ for each τ such that P_τ is an associated prime. But since I_k is the intersection of the $I_{k\tau}$, where τ ranges over all associated primes P_τ of I_k, this means that $I_1 \supsetneq I_2 \supsetneq \ldots$, where the inclusions are proper, since the I_k are all distinct. $\square$

Chapter 3

Boolean Term Orders and the Root System B_n

3.1 Introduction

This chapter studies Gröbner basis term orders in the exterior algebra, which we call *Boolean term orders*.

Definition 3.1.1. A *Boolean term order* is a total order on subsets of $[n] = \{1, \ldots, n\}$ such that:

1. $\emptyset \prec \alpha$ for all $\alpha \subseteq [n], \alpha \neq \emptyset$.

2. If $\alpha \prec \beta$, and $\gamma \cap (\alpha \cup \beta) = \emptyset$ then $\alpha \cup \gamma \prec \beta \cup \gamma$

Where there can be no confusion, we will use the phrase *term order*.

Boolean term orders arise in several different areas of mathematics. My motivation is that they are Gröbner basis term orders in the exterior algebra over a vector space of dimension n. A monomial x^α in the exterior algebra corresponds to the subset of $[n]$ given by $\mathrm{supp}(\alpha) = \{i : x_i \text{ divides } x^\alpha\}$. The axioms above then correspond to the standard ones for Gröbner basis theory term orders. Gröbner basis theory for the exterior algebra is described in [41] and [2].

Another incarnation of Boolean term orders is as antisymmetric comparative probability relations. In that context is that we have a finite set Ω of events, and are interested in which events are more probable than others, as opposed to the exact probability of each event. We associate with a subset of Ω the (comparative) probability that at least one of the events in the subset occurs. If we demand that for any two subsets A and B of Ω either $A \prec B$ (B is more probable than A), or $B \prec A$, then we have a Boolean term order with $n = |\Omega|$. An overview of the theory of comparative probability can be be found in [15].

The main result of this chapter is that Boolean term orders correspond to one element extensions of the oriented matroid $\mathcal{M}(B_n)$, where B_n is the root system $\{e_i : 1 \leq i \leq n\} \cup \{e_i \pm e_j : 1 \leq i < j \leq n\}$. This establishes Boolean term orders in the frame work of the Baues problem, in the sense of [36]. Section 2 contains precise definitions of term orders, including a notion of coherence. The connection to the Baues problem is strengthened in Section 3 with the introduction of a notion of a flip relation between different Boolean term orders. Examples are given of term orders exhibiting flip deficiency. Section 4 examines the structure of the set of coherent term orders, and part of the connection to B_n. This connection is fully established in Section 5 with the proof of the main theorem described above. Finally, Section 6 contains some specific examples of Boolean term orders, and enumeration for small values of n.

3.2 Definitions

Definition 3.2.1. A *generalized partial term order* is a partial order, $\prec$, on the set of subsets of $[n]$ such that:

1. $\alpha \prec \beta \Leftrightarrow \alpha \cup \gamma \prec \beta \cup \gamma$ for all γ with $\gamma \cap (\alpha \cup \beta) = \emptyset$
2. If α and β are not comparable (written $\alpha \sim \beta$), then $\{\gamma : \gamma \prec \alpha\} = \{\gamma : \gamma \prec \beta\}$, and $\{\gamma : \alpha \prec \gamma\} = \{\gamma : \beta \prec \gamma\}$

We write $\alpha \preceq \beta$ if $\alpha \sim \beta$ or $\alpha \prec \beta$. A *partial Boolean term order* additionally satisfies $\emptyset \preceq \alpha$ for all $\alpha \neq \emptyset$. A *generalized term order* is a generalized partial term order where $\alpha \sim \beta \Rightarrow \alpha = \beta$.

Lemma 3.2.2. *1. The first condition of Definition 3.2.1 is equivalent to requiring that whenever $a \prec b$, $c \prec d$ with $a \cap c = \emptyset$ and $b \cap d = \emptyset$, then $a \cup c \prec b \cup d$*

2. Assuming the first condition of Definition 3.2.1, the second condition is equivalent to requiring that if either $a \cup c \sim b \cup d$ or $a \cup c = b \cup d$, and $b \prec a$, then $c \prec d$.

Proof. 1. Assume $\prec$ satisfies the first condition of Definition 3.2.1 and write $b = b' \cup l$, $c = c' \cup l$, where $b' \cap c' = \emptyset$. Then $a \cup c' \prec b \cup c' = b' \cup c \prec b' \cup d$, so $a \cup c' \cup l \prec b' \cup d \cup l$. This proves one implication.

Conversely, assume that $\prec$ is a total order on subsets of $[n]$ satisfying the condition of the lemma, but not satisfying the first condition of Definition 3.2.1, so there is a pair $a \prec b$ with $b \cup c \prec a \cup c$. We may assume that $a \cap b = \emptyset$, so the condition of the lemma implies $a \cup b \cup c \prec a \cup b \cup c$, a contradiction.

2. Suppose $\prec$ satisfies both conditions of Definition 3.2.1, $b \prec a$, and either $a \cup c \sim b \cup d$ or $a \cup c = b \cup d$. If $d \prec c$ we would have $b \cup d \prec a \cup c$ from above. If $c \sim d$, then write $b = b' \cup l$, $c = c' \cup l$ as above. Then $b' \cup c = b \cup c' \prec a \cup c'$. But $b' \cup c \sim b' \cup d$, so $b' \cup d \prec a \cup c'$. From this contradiction we conclude $c \prec d$.

Conversely, let $\prec$ be a partial order on monomials satisfying the first condition of Definition 3.2.1, and such that whenever $b \prec a$ and either $a \cup c \sim b \cup d$ or $a \cup c = b \cup d$ then $c \prec d$. Suppose $\alpha \sim \beta$, and $\alpha \prec \gamma$. We can assume that $\alpha \cap \beta \cap \gamma = \emptyset$. Write

$$\begin{aligned} \alpha &= \alpha' \cup \delta \cup \phi \\ \beta &= \beta' \cup \delta \cup \psi \\ \gamma &= \gamma' \cup \phi \cup \psi \end{aligned}$$

where $\alpha', \beta', \gamma', \delta, \phi, \psi$ are all disjoint. Then $\alpha \sim \beta$ implies $\alpha \cup \gamma' \sim \beta \cup \gamma'$. Writing this out in full, we have $(\gamma' \cup \phi) \cup (\alpha' \cup \delta) \sim (\gamma' \cup \psi) \cup (\beta' \cup \delta)$. But $\alpha \prec \gamma$ means $\alpha' \cup \delta \prec \gamma' \cup \psi$, so $\beta' \cup \delta \prec \gamma' \cup \phi$, and thus, multiplying by ψ, we get $\beta \prec \gamma$, as required by the definition. The case with $\gamma \prec \alpha$ is the same with the inequalities reversed.

□

Corollary 3.2.3. *If $\prec$ is a generalized partial term order with $a \prec b$ then $[n] \setminus b \prec [n] \setminus a$.*

Proof. We have $[n] = a \cup ([n] \setminus a)$, and $[n] = b \cup ([n] \setminus b)$, so the result follows from part 2 of Lemma 3.2.2. □

This means that the second half of the term order is the complement of the first, in reverse order.

Definition 3.2.4. A Boolean term order is *coherent* if there is a weight vector $w = (w_1, w_2, \ldots, w_n) \in \mathbb{N}^n$ such that

$$\alpha \prec \beta \Leftrightarrow \sum_{i \in \alpha} w_i < \sum_{j \in \beta} w_j$$

In the interpretation of Boolean term orders as Gröbner basis term orders in the exterior algebra, a term order is coherent if it can be extended to a Gröbner basis term order on all the monomials in n variables. This follows from the fact that a Gröbner basis term order in n variables can be induced up to a given finite degree by an integral weight vector (see, for example, Chapter 15 of [10]). In the comparative probability language these are the comparative probability orders which have an agreeing probability measure, and are known as additive antisymmetric comparative probability orders. The question of whether all antisymmetric comparative probability orders were additive was first raised by de Finetti in 1951 [8], and first answered in 1959 by Kraft, Pratt, and Seidenberg [27].

Example 3.2.5. A *noncoherent* Boolean term order for $n = 5$ is:

$$\begin{aligned}
&\emptyset \prec \{1\} \prec \{2\} \prec \{3\} \prec \{4\} \prec \{1,2\} \prec \{5\} \prec \{1,3\} \prec \{2,3\} \prec \\
&\{1,4\} \prec \{1,5\} \prec \{2,4\} \prec \{2,5\} \prec \{3,4\} \prec \{1,2,3\} \prec \{1,2,4\} \prec \\
&\{3,5\} \prec \{4,5\} \prec \{1,2,5\} \prec \{1,3,4\} \prec \{1,3,5\} \prec \{2,3,4\} \prec \\
&\{2,3,5\} \prec \{1,4,5\} \prec \{2,4,5\} \prec \{1,2,3,4\} \prec \{3,4,5\} \prec \\
&\{1,2,3,5\} \prec \{1,2,4,5\} \prec \{1,3,4,5\} \prec \{2,3,4,5\} \prec \{1,2,3,4,5\}
\end{aligned}$$

To see that this term order is not coherent, we notice that $\{4\} \prec \{1,2\}$, $\{2,3\} \prec \{1,4\}$, $\{1,5\} \prec \{2,4\}$, and $\{1,2,4\} \prec \{3,5\}$. In the exterior algebra characterization, this is $x_4 \prec x_1x_2$, $x_2x_3 \prec x_1x_4$, $x_1x_5 \prec x_2x_4$, and $x_1x_2x_4 \prec x_3x_5$. Note that if $\prec$ extended to an order on all monomials, we could multiply all the left and right sides to get $x_1^2x_2^2x_3x_4^2x_5 \prec x_1^2x_2^2x_3x_4^2x_5$, a contradiction. So there is no order on the polynomial ring extending $\prec$.

This method of giving a certificate for the noncoherency of a Boolean term order has received attention recently in the work of Fishburn [16], [17]. An open problem is to give a sharp upper bound on the number of inequalities needed in such a certificate for term orders on subsets of $[n]$.

3.3 Flips for Term Orders

In this section we place Boolean term orders in the framework of the Baues problem. Specifically, we define a notion of flip for a term order, analogous to bistellar flips for triangulations.

Definition 3.3.1. A *primitive pair* in a term order is a pair $\alpha \prec \beta$, with $\alpha \cap \beta = \emptyset$, which is *adjacent*, in the sense that there is no γ with $\alpha \prec \gamma \prec \beta$.

Example 3.3.2. In the term order of Example 3.2.5, the primitive pairs are: $\emptyset \prec \{1\}$, $\{1\} \prec \{2\}$, $\{2\} \prec \{3\}$, $\{3\} \prec \{4\}$, $\{4\} \prec \{1,2\}$, $\{1,2\} \prec \{5\}$, $\{5\} \prec \{1,3\}$, $\{2,3\} \prec \{1,4\}$, $\{1,5\} \prec \{2,4\}$, $\{2,5\} \prec \{3,4\}$, and $\{1,2,4\} \prec \{3,5\}$.

Proposition 3.3.3. *The order on the primitive pairs determines the Boolean term order, in the sense that if $\prec_1$ and $\prec_2$ are different term orders, then there is a primitive pair of $\prec_1$, $\alpha \prec_1 \beta$, such that $\beta \prec_2 \alpha$.*

Proof. Suppose that $\prec_1$ and $\prec_2$ are two distinct term orders which have the same order on all the primitive pairs of $\prec_1$.

Name each monomial by μ_k and γ_l for some k and l, so that $i < j \Rightarrow \mu_i \prec_1 \mu_j$, and $i < j \Rightarrow \gamma_i \prec_2 \gamma_j$. Let α be the greatest monomial in order for which the two term orders agree, so that if $\alpha = \mu_k = \gamma_l$ then $k = l$, and $i < k \Rightarrow \mu_i = \gamma_i$, but $\mu_{k+1} \neq \gamma_{k+1}$. There must be such a monomial, as both orders agree that 1 is the smallest monomial. Denote the next subset for $\prec_1$, μ_{k+1}, by β, and for $\prec_2$, γ_{k+1}, by γ. By assumption $\beta \neq \gamma$.

Let δ be the subset immediately preceding γ in $\prec_1$. We know that $\delta \cap \gamma \neq \emptyset$, as otherwise $\delta \prec_1 \gamma$ is a primitive pair of $\prec_1$, so from consideration of $\prec_2$ we see that δ is at most α, which would mean $\beta = \gamma$. Denote $\delta \setminus \gamma$ and $\gamma \setminus \delta$ by δ' and γ' respectively. Looking at $\prec_2$, we see that γ' is at most α, so, since $\delta' \prec_1 \gamma'$, $\delta' \prec_2 \gamma'$, and thus $\delta \prec_2 \gamma$, contradicting the adjacency of α and γ with respect to $\prec_2$. $\square$

Definition 3.3.4. A *flippable pair* in a term order $\prec$ is a primitive pair $\alpha \prec \beta$ such that all multiples $\alpha \cup l \prec \beta \cup l$ are adjacent.

Example 3.3.5. In the term order of Example 3.2.5, the flippable pairs are $\{4\} \prec \{1,2\}$, $\{2,3\} \prec \{1,4\}$, $\{1,5\} \prec \{2,4\}$, $\{2,5\} \prec \{3,4\}$, and $\{1,2,4\} \prec \{3,5\}$.

Definition 3.3.6. Given a Boolean term order $\prec_1$ and a flippable pair $\alpha \prec_1 \beta$, with $\alpha \neq \emptyset$, we construct a new total order $\prec_2$ by exchanging the order of $\alpha \cup l$ and $\beta \cup l$ for all $l \subseteq [n] \setminus (\alpha \cup \beta)$. We say that $\prec_2$ is obtained from $\prec_1$ by *flipping across* $\alpha \prec_1 \beta$.

Proposition 3.3.7. *The $\prec_2$ constructed above is a Boolean term order.*

Proof. Since $\emptyset$ is still the smallest element in $\prec_2$, we need only check that the order satisfies the first condition of Definition 3.1.1. Suppose it does not, so there exist $a,\ b,\ c \subseteq [n]$ such that $a \prec_2 b$, but $b \cup c \prec_2 a \cup c$. The only pairs whose orders have changed from $\prec_1$ are the multiples of $\alpha \prec_1 \beta$, so one of $a \prec_2 b$, and $b \cup c \prec_2 a \cup c$ must be such a multiple. But if one pair is a multiple of $\beta \prec_2 \alpha$, so is the other, and so their order is still consistent, as it is the reverse of the consistent order under $\prec_1$. Thus $a \prec_2 b \Leftrightarrow a \cup c \prec_2 b \cup c$, so $\prec_2$ is a Boolean term order. □

Remark 3.3.8. Note that $\prec_1$ and $\prec_2$ agree on every pair of disjoint sets except for the pair $\{\alpha, \beta\}$.

Example 3.3.9. Flipping across $\{4\} \prec \{1,2\}$ in the term order of Example 3.2.5 gives:

$$\begin{aligned}&\emptyset \prec \{1\} \prec \{2\} \prec \{3\} \prec \{1,2\} \prec \{4\} \prec \{5\} \prec \{1,3\} \prec \\ &\{2,3\} \prec \{1,4\} \prec \{1,5\} \prec \{2,4\} \prec \{2,5\} \prec \{1,2,3\} \prec \\ &\{3,4\} \prec \{1,2,4\} \prec \{3,5\} \prec \dots\end{aligned}$$

Only the first half of the order is given, as the second half can be deduced from the first by Corollary 3.2.3. This is a coherent term order, given by the weight vector $(7, 10, 16, 20, 22)$.

Remark 3.3.10. The analogue of Proposition 3.3.3 for flippable pairs is false. The following noncoherent term order in six variables, with the flippable pairs marked by $\prec_*$, has the same order on its flippable pairs as the coherent term order given by the vector $(6, 14, 15, 18, 28, 38)$. It is thus not determined by the order on its flippable pairs.

$$\begin{aligned}&\emptyset \prec \{1\} \prec \{2\} \prec \{1,2\} \prec \{3\} \prec \{1,3\} \prec \{4\} \prec \{2,3\} \prec \{1,4\} \prec \\ &\{1,2,3\} \prec \{2,4\} \prec \{5\} \prec \{1,2,4\} \prec \{3,4\} \prec_* \{1,5\} \prec \{2,5\} \prec \\ &\{6\} \prec_* \{1,3,4\} \prec \{2,3,4\} \prec \{1,2,5\} \prec \{3,5\} \prec_* \{1,6\} \prec \\ &\{2,6\} \prec \{1,2,3,4\} \prec \{1,3,5\} \prec \{4,5\} \prec \{2,3,5\} \prec \{1,2,6\} \prec \\ &\{1,4,5\} \prec_* \{3,6\} \prec \{1,2,3,5\} \prec \{1,3,6\} \prec_* \{2,4,5\} \dots\end{aligned}$$

Remark 3.3.11. The central pair is flippable in every term order. By Corollary 3.2.3 the $2^{n-1}+1$st term is the complement of the support of the 2^{n-1}th term, so there are no nontrivial multiples to consider.

3.4 Hyperplane Arrangements and the Root System B_n

The equivalence classes of real weight vectors w which determine the same coherent generalized term order correspond to regions of a hyperplane arrangement. This arrangement, denoted $\mathcal{H}_n$, lies in $\mathbb{R}^n$ and consists of all the hyperplanes with normals in $\{0,1,-1\}^n \setminus \mathbf{0}^n$.

Lemma 3.4.1.
1. *The n-dimensional regions of $\mathcal{H}_n$ are in bijection with coherent generalized term orders.*
2. *Flipping across a flippable pair from one coherent term order to another coherent term order corresponds geometrically to passing from one region of $\mathcal{H}_n$ to an adjacent region.*

Proof. Part 1 is immediate from the definition of $\mathcal{H}_n$. Part 2 follows from Remark 3.3.8, since if two regions of $\mathcal{H}_n$ are on the same side of all but one hyperplane, as is the case for the regions corresponding to term orders connected by a flip, then the two regions must be adjacent. □

It would thus be interesting to know the number of regions of $\mathcal{H}_n$. Unfortunately this does not appear to be a simple combinatorial function. One way to compute the number of regions of a hyperplane arrangement is via its characteristic polynomial $\chi_{\mathcal{H}_n}(x)$, which is defined in terms of the lattice of intersections of the arrangement (see page 43 of [31] for details). A result of Zaslavsky (Theorem 2.68 in [31]) states that $|\chi_{\mathcal{H}_n}(-1)|$ is the number of regions of the arrangement. For a particular class of hyperplane arrangements, known as free arrangements, the characteristic polynomial is known to have integer roots. In general, $\mathcal{H}_n$ is not a free arrangement, as can be seen from the following table.

n	$\chi_{\mathcal{H}_n}(x)$
1	$x-1$
2	$(x-1)(x-3)$
3	$(x-1)(x-5)(x-7)$
4	$(x-1)(x-11)(x-13)(x-15)$
5	$(x-1)(x-29)(x-31)(x^2-60x+971)$
6	$(x-1)(x^5-363x^4+54310x^3-4182690x^2+165591769x-2691439347)$
7	$(x-1)(x^6-1092x^6+518385x^4-136815000x^3+21151739259x^2$
	$-1814252700708x+67379577529235)$

From part 2 of Lemma 3.4.1 we know that the number of facets of a particular region corresponds to the number of coherent neighbors the corresponding term order has. The arrangements in dimension two and three are simplicial, but this is no longer the case in dimension four.

Example 3.4.2. The following term order has five flippable pairs. From enumerative results (see Section 3.6) we know that all term orders are coherent in dimension four, so this term order corresponds to a cone with five facets. This is the maximal number of facets of a region of $\mathcal{H}_4$.

$$\begin{aligned}&\emptyset \prec \{1\} \prec_* \{2\} \prec_* \{3\} \prec_* \{1,2\} \prec \{1,3\} \prec \{2,3\}\\ &\prec_* \{4\} \prec_* \{1,2,3\} \prec \ldots\end{aligned}$$

The comparisons marked $\prec_*$ are flippable pairs.

Remark 3.4.3. Since every n dimensional region of $\mathcal{H}_n$ has at least n facets, coherent term orders have at least n flippable pairs. The same is not true for noncoherent term orders.

Theorem 3.4.4. *For $n \geq 6$ there are term orders with fewer than n flippable pairs. These are examples of* flip deficiency *in the sense of [36].*

Proof. The term order on subsets of [6] in Remark 3.3.10 of Section 3.3 has only five flippable pairs. Using this as a base, we can construct term orders on subsets of $[n]$ with $n-1$ flippable pairs by setting $[k] \prec \{k+1\}$ for $6 \leq k \leq n-1$. □

We now develop the connection between $\mathcal{H}_n$ and the root system B_n.

Definition 3.4.5. The root system B_n is the collection of vectors $\{\pm e_i : 1 \leq i \leq n\} \cup \{\pm e_i \pm e_j : 1 \leq i < j \leq n\}$ in $\mathbb{R}^n$, where e_i is the ith standard basis vector.

Lemma 3.4.6. *$\mathcal{H}_n$ is the discriminantal arrangement of B_n. In other words, $\mathcal{H}_n$ is the collection of hyperplanes $\{H : H$ is spanned by a subset of B_n $\}$.*

Proof. Let H be a hyperplane in $\mathcal{H}_n$, with normal vector v. Let $P = \{i : v_i > 0\}$, $N = \{i : v_i < 0\}$, and $Z = \{i : v_i = 0\}$. Then $H = \mathrm{span}(\{e_i | i \in Z\} \cup \{e_i - e_j | i, j \in P\} \cup \{e_i - e_j | i, j \in N\} \cup \{e_i + e_j | i \in P, j \in N\})$. For the other inclusion, let v be the normal vector of a hyperplane spanned by a subset V of B_n. For all $e_i - e_j \in V$ we have $v_i = v_j$, and for all $e_i + e_j \in V$ we have $v_i = -v_j$. Also, whenever $e_i \in V$, we have $v_i = 0$. Suppose there exists i, j with $v_i, v_j \neq 0$ such that $v_i \neq \pm v_j$. Then let w be the vector with $w_k = v_k$ when $v_k = \pm v_i$, and $w_k = 0$ otherwise, and let u be the vector with $u_k = v_k$ when $v_k = \pm v_j$ and $u_k = 0$ otherwise. Then $V \subseteq u^\perp \cap w^\perp$, so it does not span a hyperplane. $\square$

Let $\mathcal{M}(B_n)$ denote the oriented matroid of the vector configuration B_n. A wealth of information about oriented matroids can be found in [6], along with many equivalent definitions. For the purposes of the next result, we use the following chirotope definition.

Definition 3.4.7. Given a vector configuration V of $d > n$ vectors in $\mathbb{R}^n$, the *oriented matroid* $\mathcal{M}(V)$ is the function mapping ordered subsets of V, each consisting of n vectors, to $\{+, 0, -\}$, where each subset is mapped to the sign of its determinant. We identify the function with the sign vector of length $n!\binom{d}{n}$ encoding the image of this map.

We say an oriented matroid $\mathcal{M}(V)$ is *projectively unique* if whenever $\mathcal{M}(W) = \mathcal{M}(V)$ for some vector configuration W in $\mathbb{R}^d$ there is a *projective transformation* (linear transformation plus scaling individual vectors) taking W to V.

Lemma 3.4.8. *$\mathcal{M}(B_n)$ is projectively unique.*

Proof. Suppose $\mathcal{M}(V) = \mathcal{M}(B_n)$. We denote the vectors of V by v_e, where e is the vector in B_n corresponding to v_e.

We first apply a linear transformation of V to move v_{e_i} to e_i for $1 \leq i \leq n$. Note that for any vector $e \in B_n$ the determinant of the n vectors $\{v_e\} \cup \{e_i : i \neq j\}$ is $(-1)^{j-1}(v_e)_j$, while the determinant of the n vectors $\{e\} \cup \{e_i : i \neq j\}$ is $(-1)^{j-1}(e)_j$. So v_e is zero in exactly the same coordinates as e, and has the same sign in its nonzero coordinates.

We can now do further combinations of linear transformations and scaling of vectors to also move $v_{e_i+e_{i+1}}$ to $e_i + e_{i+1}$, for $1 \leq i \leq n-1$. Lastly, we scale the other vectors

$v_{e_i \pm e_j}$ so they are of the form $e_i \pm a e_j$ for some a depending on i, j and the sign. The lemma will follow if we can now show that in each case $a = 1$.

We first show this for $v_{e_i+e_j}$ when $j-i$ is odd, and $v_{e_i-e_j}$ when $j-i$ is even. Consider $v_{e_i+e_j} = e_i + a e_j$, with $j-i$ odd. The determinant of $\{e_l : l < i\} \cup \{e_l + e_{l+1} : i \le l \le j-1\} \cup \{v_{e_i+e_j}\} \cup \{e_l : l > j\}$, which should be zero, is $a-1$, so we see that $v_{e_i+e_j} = e_i + e_j$. Considering the analogous determinant for $v_{e_i-e_j}$, where $j-i$ is even we conclude that $v_{e_i-e_j} = e_i - e_j$ for these i and j.

Now consider the determinant of $\{v_{e_1-e_2}, v_{e_2+e_3}, v_{e_1+e_3}\} \cup \{e_i : 4 \le i \le n\}$, which should be zero. If we write $v_{e_1-e_2} = e_1 - a e_2$, and $v_{e_1+e_3} = e_1 + b e_3$, then this determinant is $b-a$, so $b = a$. Notice that we get the same result if we switch replace $v_{e_1-e_2}$ and $v_{e_2+e_3}$ by $v_{e_1+e_2}$ and $v_{e_2-e_3}$. Considering the analogous subdeterminants with similar adjacent subdeterminants, we can conclude that there exists a single a such that $v_{e_i-e_{i+1}} = e_i - a e_{i+1}$ for all $1 \le i \le n-1$. The determinant of $\{v_{e_1-e_3}, v_{e_1-e_2}, v_{e_2-e_3}\} \cup \{e_i : 4 \le i \le n\}$, which again should be zero, is now $a^2 - 1$, so see that $a = \pm 1$. The case $a = -1$ is ruled out by consideration of the determinant $\{v_{e_1+e_2}, v_{e_1-e_2}\} \cup \{e_i : 3 \le i \le n\}$, which is $-a-1$, and should be negative, so $a = 1$, and thus $v_{e_i-e_{i+1}} = e_i - e_{i+1}$ for $1 \le i \le n-1$.

From consideration, when $j-i$ is even, of the determinant of $\{e_l : l < i\} \cup \{v_{e_i-e_{i+1}}\} \cup \{v_{e_l+e_{l+1}} : i+1 \le l \le j-1\} \cup \{v_{e_i+e_j}\} \cup \{e_l : l > j\}$, which should be zero, but is $a-1$, we see that $v_{e_i+e_j} = e_i + e_j$. Finally, the analogous determinant for $v_{e_i-e_j}$, when $j-i$ is odd, yields $v_{e_i-e_j} = e_i - e_j$.

Thus each vector $v_e \in V$ has been moved to the corresponding vector e, so there is a projective transformation moving V to B_n, and thus we conclude that $\mathcal{M}(B_n)$ is projectively unique. □

Corollary 3.4.9. *There is a bijection between realizable one element extensions of $\mathcal{M}(B_n)$ and coherent generalized partial term orders.*

Proof. The implication of Lemma 3.4.8 is that realizable one element extensions of $\mathcal{M}(B_n)$ correspond exactly to regions of the discriminantal arrangement of B_n. Thus from Lemma 3.4.6 we get a bijection between realizable one element extensions of $\mathcal{M}(B_n)$ and regions of $\mathcal{H}_n$, and so Lemma 3.4.1 gives a bijection between realizable one element extensions of $\mathcal{M}(B_n)$ and coherent generalized partial term orders. □

3.5 Oriented Matroids

In the previous section we saw that coherent generalized partial term orders correspond to realizable one element extension of $\mathcal{M}(B_n)$. In this section we expand on this, showing that all generalized partial term orders correspond to one element extensions of $\mathcal{M}(B_n)$. We assume more familiarity with oriented matroids.

The content of Lemma 3.4.6 was that the cocircuits of $\mathcal{M}(B_n)$ are in bijection with the hyperplane normals in the set $\{+1, 0, -1\}^n \setminus \mathbf{0}^n$. We will represent these normals by their sign vectors, which will be denoted by capital letters, such as X. The corresponding cocircuit will be denoted by $\overline{X}$. The passage from X to $\overline{X}$ is as follows: $\overline{X}_{e_i} = X_i$, $\overline{X}_{e_i+e_j} = X_i + X_j$, and $\overline{X}_{e_i-e_j} = X_i - X_j$, where $+$ and $-$ applied to $\{+, 0, -\}$ evaluate to the sign of the corresponding operation on $\{+1, 0, -1\}$.

For example, if $n = 3$, and $X = (++0)$, then $\overline{X} = (++0+++0++)$ where the coordinates of $\overline{X}$ are listed in the following order: the e_i, then the $e_i + e_j$ in lexicographic order, and finally the $e_i - e_j$, also in lexicographic order.

Definition 3.5.1. X^+ is the set $\{i : X_i = +\}$. X^- is the set $\{i : X_i = -\}$. Given two sign vectors X and Y we say that a sign vector Z is an *elimination candidate* for X and Y if $\overline{Z}^+ \subseteq \overline{X}^+ \cup \overline{Y}^+$ and $\overline{Z}^- \subseteq \overline{X}^- \cup \overline{Y}^-$

We first describe some Z which are elimination candidates for given X and Y. We can decompose $X^\pm, Y^\pm$ by writing

$$\begin{aligned} X^- &= a \cup m \cup x \\ X^+ &= b \cup p \cup y \\ Y^- &= c \cup m \cup y \\ Y^+ &= d \cup p \cup x \end{aligned}$$

where a, b, c, d, m, p, x and y are all pairwise disjoint.

Lemma 3.5.2. *With decomposition as above, the following pairs (Z^+, Z^-) are elimination* *candidates for X and Y:*

$$(p, m)$$
$$(b \cup p, a \cup m)$$
$$(d \cup p, c \cup m)$$
$$(b \cup d \cup p \cup y, a \cup c \cup m \cup x)$$
$$(b \cup d \cup p \cup x, a \cup c \cup m \cup y)$$

In addition, if $m = p = \emptyset$, then $(b \cup d, a \cup c)$ is also an elimination candidate.

Proof. We can encode each of these vectors by a sign vector S listing the sign of Z_i for $i \in b, d, p,$ and x. For example, (p, m) can be encoded as $(0, 0, +, 0)$. This coding is reversible, as for all these pairs (Z^+, Z^-), and also for $(Z^+, Z^-) \in \{(X^+, X^-), (Y^+, Y^-)\}$, we have $(b \cup d \cup p) \cap Z^- = \emptyset$, and $A \subseteq Z^+ \Rightarrow B \subseteq Z^-$, for $(A, B) \in \{(b, a), (c, d), (p, m), (x, y), (y, x)\}$. In this example, when we pass to $\overline{(p, m)}$, this can be encoded as $(0, 0, +, 0, 0, +, 0, +, 0, +, 0, -, 0, -, 0, +)$, with the coordinates ordered first b, d, p, x, then sums in lexicographic order, and finally differences in lexicographic order. The elimination candidate condition is then equivalent to requiring that, under this encoding, whenever $\overline{S}_e = +$, either $\overline{X}_e = +$ or $\overline{Y}_e = +$, and similarly whenever $\overline{S}_e = -$ either $\overline{X}_e = -$ or $\overline{Y}_e = -$.

This is easy to check. We have the following encoding of X and Y:

$$\begin{aligned} \overline{X} &= (+, 0, +, -, +, +, 0, +, -, 0, +, 0, +, -, +, +) \\ \overline{Y} &= (0, +, +, +, +, +, +, +, +, +, -, -, -, 0, 0, 0) \end{aligned}$$

From this we can see that (p, m) is an elimination candidate. The rest of the proof is listing the remaining four vectors.

In the case where $m = p = \emptyset$, we have the reduced encoding of X and Y as $(+, 0, -)$ and $(0, +, +)$, leaving out the pth coordinate. The condition to be an elimination candidate remains the same, so from the encoding of $(b \cup d, a \cup c)$ as $(+, +, 0, +, +, +, 0, +, +)$ we can see that it is an elimination candidate. ☐

We are now ready for the main theorem. A generalized partial term order $\prec$ induces a function μ from $\{+, 0, -\}^n$ to $\{+, 0, -\}$ by

$$\mu(X) = \begin{cases} + & \text{if } X^- \prec X^+ \\ - & \text{if } X^+ \prec X^- \\ 0 & \text{otherwise} \end{cases}$$

Let σ be the induced function on the cocircuits of $\mathcal{M}(B_n)$ given by $\sigma(\overline{X}) = \mu(X)$.

Theorem 3.5.3. *The map σ is a localization, which gives a map from the set of generalized partial term orders to the set of one element extensions of $\mathcal{M}(B_n)$. This map is an injection.*

Proof. Since different generalized partial term orders give different functions μ, it suffices to prove that σ is a localization. By Corollary 7.1.9 of [6], this is equivalent to showing that the set $\sigma^{-1}(\{+,0\})$ satisfies the weak cocircuit elimination axiom. This says that for any X,Y with $\overline{X} \neq -\overline{Y}$ and $\mu(X), \mu(Y) \in \{+,0\}$ such that $\overline{X}_e = +$ and $\overline{Y}_e = -$ for some $e \in B_n$, there is some Z with $\overline{Z}_e = 0$ and $\mu(Z) \in \{+,0\}$ which is an elimination candidate for X and Y.

We first show that if $\mu(X), \mu(Y) \in \{+,0\}$ then there is an elimination candidate, Z, for X and Y with $\mu(\overline{Z}) \in \{+,0\}$. We decompose $X^{\pm}, Y^{\pm}$ as above, and write the candidate Z as (Z^+, Z^-). Parentheses will be omitted at times to simplify notation. Note that the condition that $\mu(X), \mu(Y) \in \{+,0\}$ means $a \cup m \cup x \preceq b \cup p \cup y$ and $c \cup m \cup y \preceq d \cup p \cup x$, and also that $\overline{X} \neq -\overline{Y}$. The argument divides into two cases. We make repeated use of Lemma 3.2.2.

Case I: $m \preceq p$. Then $\mu(p, m) \in \{+,0\}$.

Case II: $p \prec m$ or $m = p = \emptyset$. Then $a \cup x \preceq b \cup y$, and $c \cup y \preceq d \cup x$. There are three further cases.

Case (a): $x \prec y$. Then $c \cup m \prec d \cup p$, so $\mu(d \cup p, c \cup m) = +$. Also $c \prec d$, so $\mu(b \cup d \cup p \cup y, a \cup c \cup m \cup x) = +$. Note that these are both nonempty pairs in the case $m = p = \emptyset$.

Case (b): $y \prec x$. Then $\mu(b \cup p, a \cup m) = +$ and $a \prec b$, so $\mu(b \cup d \cup p \cup x, a \cup c \cup m \cup y) = +$. Again, these are both nonempty pairs.

Case (c): $x \sim y$ or $x = y = \emptyset$. We break into two further cases.

Case (i): $p \prec m$. Then $a \cup m \preceq b \cup p$, so $\mu(b \cup p, a \cup m) \in \{+,0\}$, and similarly $\mu(d \cup p, c \cup m) \in \{+,0\}$. Also $c \prec d$, so $\mu(b \cup d \cup p \cup y, a \cup c \cup m \cup x) = +$.

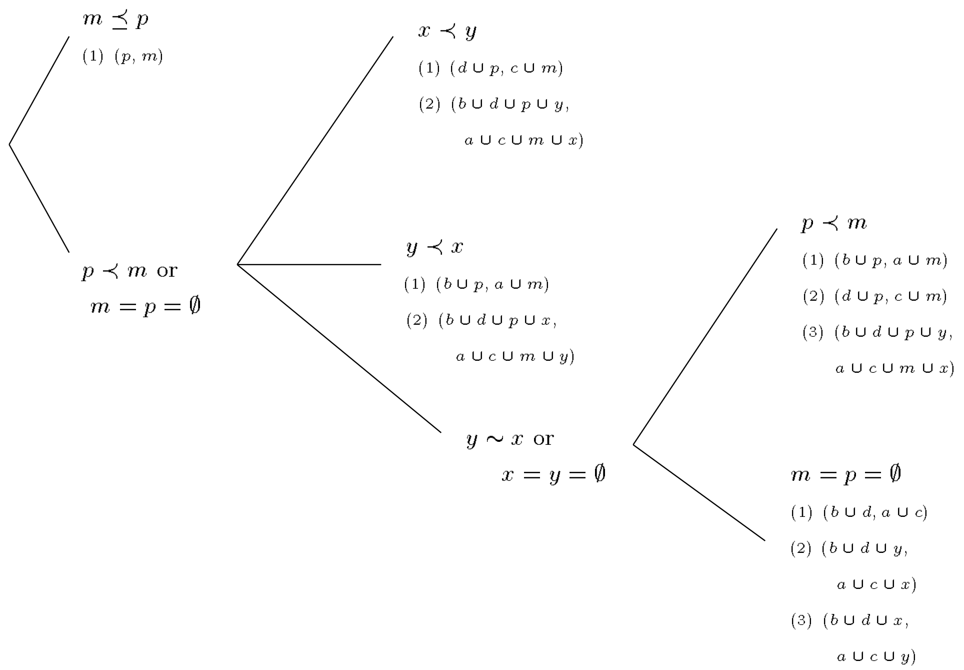

Figure 3.1: μ-nonnegative vectors

Case (ii): $m = p = \emptyset$. Then $\mu(b \cup d, a \cup c), \mu(b \cup d \cup y, a \cup c \cup x), \mu(b \cup d \cup x, a \cup c \cup y) \in \{+, 0\}$. Since $\overline{X} \neq \overline{Y}$, these are all nonempty pairs.

So we have the diagram of cases shown in Figure 1. Note that in each branch the pair is nonempty, so it does represent a cocircuit. By Lemma 3.5.2 the pair in each branch is an elimination candidate for X and Y. So we have shown that if $\mu(X), \mu(Y) \in \{+, 0\}$ then there is an elimination candidate, Z, for X and Y with $\mu(Z) \in \{+, 0\}$. We are now ready to show that $\sigma^{-1}(\{+, 0\})$ satisfies the weak cocircuit elimination condition. Recall that this involves showing for all $X, Y \in \mu^{-1}(\{+, 0\})$ with $\overline{X} \neq -\overline{Y}$, and $e \in B_n$ such that $\overline{X}_e = + = -\overline{Y}_e$ there exists a $Z \in \mu^{-1}(\{+, 0\})$ which is an elimination candidate for X and Y with $\overline{Z}_e = 0$. To do this, it suffices to show that if $\overline{X}_e = +$ and $\overline{Y}_e = -$ then on every branch of Figure 1 there is a Z with $\overline{Z}_e = 0$. This is again a consideration of cases, depending on the form of e. The following tables enumerate the cases, with the rightmost column indicating which Z is chosen for each branch of the above diagram. We read the nodes of the diagram in the order the corresponding cases appear above; that is, top to bottom then left to right. The numbers are those in the diagram, so 11111 represents the choice of (p, m), $(d \cup p, c \cup m)$, $(b \cup p, a \cup m)$, $(b \cup p, a \cup m)$, and $(b \cup d, a \cup c)$ in the corresponding branches.

1. $e = e_i$. Then $i \in x \cup y$ and we require $Z_i = 0$, so we take 11111.

2. $e = e_i + e_j$. Then we require $Z_i = -Z_j$

X_i	X_j	Y_i	Y_j	$i \in$	$j \in$	Z
+	+	−	−	y	y	11111
+	+	−	0	y	b	11223
+	+	0	−	b	y	11223
+	0	−	−	y	c	12112
+	0	−	0	y	0	11111
+	0	0	−	b	c	12231
0	+	−	−	c	y	12112
0	+	−	0	c	b	12231
0	+	0	−	0	y	11111

3. $e = e_i - e_j$. Then we require $Z_i = Z_j$.

X_i	X_j	Y_i	Y_j	$i \in$	$j \in$	Z
+	−	−	+	y	x	11111
+	−	−	0	y	a	11223
+	−	0	+	b	x	11223
+	0	−	+	y	d	12112
+	0	−	0	y	0	11111
+	0	0	+	b	d	12231
0	−	−	+	c	x	12112
0	−	−	0	c	a	12231
0	−	0	+	0	x	11111

This shows that $\sigma^{-1}(\{+, 0\})$ satisfies the weak cocircuit elimination axiom, so σ is a localization. □

Corollary 3.5.4. *A noncoherent generalized partial term order determines a nonrealizable one element extension of $\mathcal{M}(B_n)$.*

Proof. This follows directly from Theorem 3.5.3 and Lemma 3.4.8. □

We will show in Remark 3.5.6 that there are one element extensions of $\mathcal{M}(B_n)$ which are not induced by a generalized partial term order in this way. To this end we characterize those one element extensions which are in the image of the map from generalized partial term orders.

Proposition 3.5.5. *Let μ be a function from $\{+,-,0\}^n \setminus \mathbf{0}^n$ to $\{+,0,-\}$. It is induced by a generalized partial term order if and only if it satisfies the following criteria:*

1. $\mu(-x) = -\mu(x)$

2. **(First Addition Condition)** *If $\mu(x) = \mu(y) = +$, and $x_i \neq y_i$ whenever $x_i \neq 0$ for $1 \leq i \leq n$, then $\mu(z) = +$, where*

$$z_i = \begin{cases} x_i & y_i = 0 \\ y_i & x_i = 0 \\ 0 & x_i = -y_i \neq 0 \end{cases}$$

3. **(Second Addition Condition)** *If $\mu(x) = 0$, $\mu(y) = +$, and $x_i \neq y_i$ whenever $x_i \neq 0$ for $1 \leq i \leq n$, then $\mu(z) = +$, where z is as above.*

Proof. The necessity of the first condition is immediate from the way μ is induced by a term order. Lemma 3.2.2 implies the second and third conditions.

For sufficiency, let μ be a function satisfying the above conditions. Given two subsets of $[n]$, define their order to be the one induced on the pair of subsets obtained by removing their intersection from each subset. Suppose that this does not define a partial order on the set of all subsets of $[n]$. Then there is some string of inequalities $b_0 \prec b_1 \prec b_2 \prec \ldots \prec b_n \preceq b_0$. We can reduce this to a string of three distinct monomials. If $b_i \prec b_{i+2}$, we can remove b_{i+1} from the chain and repeat this procedure with the shorter chain. Otherwise we have the three element chain $b_i \prec b_{i+1} \prec b_{i+2} \preceq b_i$. Denote this three element chain $\alpha \prec \beta \prec \gamma \preceq \alpha$. We may assume that $\alpha \cap \beta \cap \gamma = \emptyset$. Write

$$\begin{aligned} \alpha &= \alpha' \cup x \cup y \\ \beta &= \beta' \cup x \cup z \\ \gamma &= \gamma' \cup y \cup z \end{aligned}$$

where α', β' and γ' are pairwise disjoint. Then we have $\alpha' \cup y \prec \beta' \cup z$, $\beta' \cup x \prec \gamma' \cup y$, and $\gamma' \cup z \preceq \alpha' \cup x$. But the addition condition demands $\alpha' \cup x \prec \gamma' \cup z$ so μ determines a partial order on subsets of $[n]$.

Suppose the partial order does not satisfy the condition on incomparable elements, so there is $\alpha \sim \beta$, $\alpha \prec \gamma$, but $\gamma \preceq \beta$. Decompose α, β, and γ as above. Then $\alpha' \cup y \sim \beta' \cup z$, and $\alpha' \cup x \prec \gamma' \cup z$ but $\gamma' \cup y \preceq \beta' \cup x$. Then $\mu(\alpha' \cup y, \beta' \cup z) = 0$, $\mu(\gamma' \cup z, \alpha' \cup x) = +$, and $\mu(\gamma' \cup y, \beta' \cup x) \in \{0, -\}$. However the second addition condition demands $\mu(\gamma' \cup y, \beta' \cup x) = +$, so the order satisfies the condition on incomparable elements.

Since the partial order satisfies the first condition of Definition 3.2.1 (the multiplicative condition) by construction, it is a generalized partial term order, and so μ is induced by a generalized partial term order. □

Remark 3.5.6. The following list of hyperplanes in $\mathbb{R}^3$ satisfies the weak cocircuit elimination axiom when considered as cocircuits of B_3:

$$(-+0), (0-+), (+0-), (+00), (0+0), (00+), (0++),$$
$$(+0+), (++0), (+++), (++-), (+-+), (-++)$$

Thus when we set $\sigma(\overline{X}) = +$ for each hyperplane X, this determines a one element extension of $\mathcal{M}(B_3)$. There is no term order, however, which induces this σ, as the set does not satisfy the first addition condition. Explicitly, if these were the positive cocircuits from some term order, then from the first three we have $x \prec y$, $y \prec z$, and $z \prec x$, which is a contradiction. So not all one element extensions of $\mathcal{M}(B_n)$ are induced by generalized partial term orders.

3.6 Numerical Results and Examples

In [14] Fine and Gill give crude bounds on the number of antisymmetric comparative probability relations, and the number of antisymmetric additive comparative probability relations, and the first terms of each sequence.

With improved computer speeds it is now possible to evaluate a few more terms in each sequence. The values calculated are displayed in the following table. The numbers in the second and third column are divided by $n!$, taking into account the action of the symmetric group.

n	Number of term orders/$n!$	Number of coherent term orders/$n!$
1	1	1
2	1	1
3	2	2
4	14	14
5	546	516
6	169444	124187
7	560043206	214580603

These were calculated with the aid of programs written by Michael Kleber and Josh Levenberg. To calculate the total number of term orders we used a recursive procedure, calculating for each term order on subsets of $[n-1]$ the number of ways the subsets involving n could be shuffled in. To calculate the number of coherent term orders we enumerated the regions of the corresponding hyperplane arrangement $\mathcal{H}_n$. The number of coherents has been also checked for $n \leq 6$ by enumerating all term orders, and counting the number of coherent ones (which can be checked by a linear program).

The following table enumerates the number of flippable pairs each term order has for orders on subsets of $\{1, 2, 3, 4, 5, 6\}$.

k	Number with k flippable pairs
5	107
6	14699
7	46626
8	56707
9	35555
10	12763
11	2633
12	334
13	20
Total	169444

Notice that there are some orders with significantly more than six flippable pairs.

Definition 3.6.1. The *Baues poset* for term orders has as its elements all generalized partial term orders. The order relation is that a term order $\prec_1$ is less than another term order $\prec_2$ if $\prec_1$ is a refinement of $\prec_2$. This poset has a $\hat{1}$, the generalized partial term order with all subsets unrelated.

Proposition 3.6.2. *There exists a noncoherent Boolean term order which lies below no coherent partial term order in the Baues poset except* $\hat{1}$.

Proof. The following order is such an example:

$$\begin{aligned}&\emptyset \prec \{1\} \prec \{2\} \prec \{1,2\} \prec \{3\} \prec \{1,3\} \prec \{2,3\} \prec \{1,2,3\} \prec \\ &\{4\} \prec \{1,4\} \prec \{2,4\} \prec \{1,2,4\} \prec \{3,4\} \prec \{5\} \prec \{1,3,4\} \prec \\ &\{2,3,4\} \prec \{1,5\} \prec \{2,5\} \prec \{1,2,3,4\} \prec \{1,2,5\} \prec \{3,5\} \prec \\ &\{1,3,5\} \prec \{2,3,5\} \prec \{6\} \prec \{1,2,3,5\} \prec \{1,6\} \prec \{4,5\} \prec \\ &\{1,4,5\} \prec \{2,6\} \prec \{1,2,6\} \prec \{3,6\} \prec \{1,3,6\} \ldots\end{aligned}$$

If it were less than a coherent partial term order other than $\hat{1}$ in the Baues poset, the linear program obtained by seeking a weight vector with all $\prec$ above relaxed to $\preceq$ would have a nonzero solution. This is not the case. ☐

Note that the example in the proof above is adjacent to a coherent order, however. Flipping over $\{5\} \prec \{1,3,4\}$ yields the coherent order given by the vector $(2,9,12,28,48,70)$.

The following term order in six variables, with flippable pairs marked with $\prec_*$, is not adjacent to any coherent term order:

$$\begin{aligned}&\emptyset \prec_* \{1\} \prec \{2\} \prec \{1,2\} \prec \{3\} \prec \{1,3\} \prec \{4\} \prec \{1,4\} \prec \{5\} \prec \\ &\{1,5\} \prec *\{2,3\} \prec \{1,2,3\} \prec \{6\} \prec \{1,6\} \prec \{2,4\} \prec \{1,2,4\} \prec \\ &\{3,4\} \prec \{1,3,4\} \prec_* \{2,5\} \prec \{1,2,5\} \prec \{2,6\} \prec \{1,2,6\} \prec_* \\ &\{3,5\} \prec \{1,3,5\} \prec \{3,6\} \prec \{1,3,6\} \prec_* \{4,5\} \prec \{1,4,5\} \prec \\ &\{2,3,4\} \prec \{1,2,3,4\} \prec \{2,3,5\} \prec \{1,2,3,5\} \prec_* \{4,6\} \ldots\end{aligned}$$

This is the case because $\{5\} \prec \{2,3\}$, $\{3,4\} \prec \{2,5\}$, $\{2,6\} \prec \{3,5\}$, and $\{2,3,5\} \prec \{4,6\}$, which, in monomial notation, gives $x_2^2x_3^2x_4x_5^2x_6 \prec x_2^2x_3^2x_4x_5^2x_6$, implying that $\prec$ is noncoherent. Flipping across any of the flippable pairs does not change any of these four inequalities, so none of the neighbors of $\prec$ are coherent.

This term order was constructed by taking a "lexicographic product" of the noncoherent term order of Example 3.2.5 with $\emptyset \prec \{1\}$. Using this product construction we can construct Boolean term orders in n variables which are at least 2^{n-5} flips from a coherent Boolean term order.

3.7 Questions

The following questions are natural in the context of a Baues problem.

1. Is the space of term orders connected by flips? This has been experimentally verified for $n \leq 6$.

2. What is the homotopy type of the poset of (generalized) partial term orders? In particular, is it spherical? The subposet of coherent generalized partial term orders is easily seen to be spherical.

3. What is the limit of the ratio of the number of coherent term orders to the total number of term orders as n increases? Is it zero?

4. What is an upper bound for the number of coherent neighbors for a coherent term order? In other words, how many facets do regions of the hyperplane arrangement have?

5. Can we give a lower bound for the total number of flippable pairs in all term orders? A good upper bound?

The final question comes from the interpretation of Boolean term orders as Gröbner basis term orders in the exterior algebra.

6. Does there exist a noncoherent initial ideal of an ideal I in the exterior algebra? That is, is there some initial ideal of I with respect to some noncoherent term order which is not equal to the initial ideal of I with respect to any coherent term order?

Chapter 4

Gröbner Bases in Monomial Quotients of a Polynomial Ring

4.1 Gröbner Basics

In this chapter we consider the problem of Gröbner basis theory in a quotient of a polynomial ring by a monomial ideal. As was the case in the last chapter when considering Gröbner bases in the exterior algebra, it is often the case that not all term orders on these quotients descend from term orders on the polynomial ring. We present answers in this framework to analogues of questions 3 and 6 from 3.7 of the previous chapter.

Let I be an Artinian monomial ideal in the polynomial ring $S = k[x_1, \ldots, x_n]$. Then S/I is a finite dimensional k vector space with basis the set of images of monomials of S not in I, which we denote by M. We are concerned with particular orderings on M.

Definition 4.1.1. A *term order*, $\prec$, on M (also called a term order on S/I) is a total order on M which satisfies

1. $1 \prec x^\alpha$ for all $\alpha \neq 0$ in M.
2. $x^\alpha \prec x^\beta \Rightarrow x^{\alpha+\gamma} \prec x^{\beta+\gamma}$ whenever $x^{\alpha+\gamma}$ and $x^{\beta+\gamma}$ are both in M.

A Boolean term order (see Chapter 3) is a term order on S/I when I is generated by the squares of the variables.

We now outline Gröbner basis theory for S/I. This is very similar to that on the exterior algebra. A set $\{g_1, \ldots, g_l\}$ of polynomials in S/I is a Gröbner basis for an ideal $J \subseteq S/I$ with respect to a term order $\prec$ if $in_\prec(G) = \{in_\prec(g_1), \dots, in_\prec(g_l)\}$ generates $in_\prec(J)$. We use the same notion of the division algorithm and reduction to zero as in the polynomial case (see, for example, [7]).

Definition 4.1.2. Let $m_{ij} = \gcd(in_\prec(g_i), in_\prec(g_j))$. An *S-polynomial*, $S(g_i, g_j)$ is the polynomial $m_{ij}/in_\prec(g_i)g_i - m_{ij}/in_\prec(g_j)g_j$, regarded as a polynomial in S/I (so the multipliers may be zero modulo I). A *T-polynomial* $T(g_i, x^\alpha)$, where $x^\alpha in_\prec(g_i) \in I$, is the polynomial $x^\alpha g_i$.

Lemma 4.1.3. *Let $h = \sum_i c_i g_i$, where for all i we have $c_i \in k$, and $in_\prec(g_i) = m$. If $in_\prec(h) \neq m$, then h is a linear combination, with coefficients in k, of the S-polynomials $S(g_i, g_j)$.*

Proof. This is word-for-word the same as the proof of Lemma 5 in Chapter 6, Section 2 of [7]. □

Theorem 4.1.4. *A set of polynomials $G = \{g_1, \dots, g_n\} \subseteq S/I$ is a Gröbner basis for the ideal $\widetilde{G}$ they generate if and only if all S-polynomials and T-polynomials reduce to zero with respect to G.*

Proof. The only-if direction is straightforward, as all S-polynomials and T-polynomials lie in $\widetilde{G}$, so reduce to zero on division by elements of a Gröbner basis.

For the if direction, suppose that all S and T polynomials of elements of G reduce to zero on division by G, and that f is a polynomial in $\widetilde{G}$. We need to show that $in_\prec(f) \in \langle in_\prec(G)\rangle$. Let $f = \sum_i h_i g_i$, where $h_i \in S$.

We first consider the case $in_\prec(h_i g_i) \neq in_\prec(h_i) in_\prec(g_i)$ for some i. This means that $in_\prec(h_i) in_\prec(g_i) \in I$, so is 0 in M. Now by assumption $T(g_i, in_\prec(h_i))$ reduces to zero on division by G. This means that there is some expression $T(g_i, in_\prec(h_i)) = \sum_j d_j m_j g_j$ where $d_j \in k$, m_j is a monomial, the g_j may be repeated, $m_j in_\prec(g_j) \notin I$, and $\max_j(m_j in_\prec(g_j)) = in_\prec(T(g_i, in_\prec(h_i)))$. We can substitute this into the expression for $h_i g_i$ to get $h_i g_i = \sum_j d_j m_j g_j + (h_i - in_\prec(h_i))g_j$, and repeat this procedure if necessary with the next term of h_i.

In this way we can get an expression $f = \sum_i h_i g_i$ where $in_\prec(h_i g_i) = in_\prec(h_i) in_\prec(g_i) \notin I$ for all i. Suppose that $\delta = \max_i(in_\prec(h_i g_i))$ is minimal over all such expressions, where the maximum is taken with respect to $\prec$. If $\delta = in_\prec(f)$, then $in_\prec(f) = in_\prec(h_i) in_\prec(g_i)$ for some i, and thus $in_\prec(f) \in \langle in_\prec(G)\rangle$. We therefore assume that $in_\prec(f) \prec \delta$.

Let $J = \{i : in_{\prec}(h_i)in_{\prec}(g_i) = \delta\}$. We use the notation $lc(f)$ for the coefficient of the leading term of f. Since $in_{\prec}(f) \prec \delta$, we must have $\sum_{j\in J} lc(h_i g_i) in_{\prec}(h_i) in_{\prec}(g_i) = 0$, so $\sum_{j\in J} lc(h_i g_i) in_{\prec}(h_i) g_i$ has initial term less than δ. This puts us in the situation of Lemma 4.1.3, so we can write $\sum_{j\in J} lc(h_i g_i) in_{\prec}(h_i) g_i = \sum_{i,j\in J} c_{ij} S(g_i, g_j)$, where $in_{\prec}(S(g_i, g_j)) \prec \delta$ for each i, j. But each $S(g_i, g_j)$ reduces to zero on division by G, so we can write it as $S(g_i, g_j) = \sum_k c_k m_k g_k$, where the g_k may be repeated, $m_k in_{\prec}(g_k) \notin I$, and $m_k in_{\prec}(g_k) \preceq in_{\prec}(S(g_i, g_j))$. Substituting this into the expression for $\sum_{j\in J} lc(h_i g_i) in_{\prec}(h_i)(g_i)$ we get an expression for f as $\sum_l p_l g_l$ where $p_l \in S$, $in_{\prec}(p_l) in_{\prec}(g_l) \notin I$, and $in_{\prec}(p_l) in_{\prec}(g_l) \prec \delta$ for all l. But this contradicts the minimality of δ, so we conclude that $\delta = in_{\prec}(f)$, and so $in_{\prec}(f) \in in_{\prec}(G)$ as required. □

We note that, as in the polynomial ring, it is not necessary to check that all S and T polynomials reduce to zero. A discussion of some of these issues for the exterior algebra is found in [41].

We now answer analogues of questions 3 and 6 from Section 3.7 of Chapter 3. This requires the following definition which is very similar in flavor to Definition 3.2.4 from Chapter 3.

Definition 4.1.5. A term order $\prec$ on M is *coherent* if there is a term order on the monomials in S which restricts to $\prec$ on M.

Example 3.2.5 of Chapter 3 is an example of a noncoherent term order for $I = \langle x_1^2, \ldots, x_n^2 \rangle$. We note that if $\prec$ is a coherent term order then Theorem 4.1.4 can be proved by considering the ideal $\widetilde{G} + I \subseteq S$. Then the required T-polynomials are S-polynomials of elements of G with monomials in I.

Definition 4.1.6. A *flippable pair* in a term order on M is a pair $x^\alpha \prec x^\beta$ with x^α and x^β relatively prime that is adjacent, in the sense that there is no x^γ with $x^\alpha \prec x^\gamma \prec x^\beta$, and all of whose multiples $x^{\alpha+\delta} \prec x^{\beta+\delta}$ are adjacent in the same sense.

Proposition 4.1.7. *Given a term order $\prec$ on M, and a flippable pair $x^\alpha \prec x^\beta$ there is a term order $\prec_2$ which agrees with $\prec$ except for pairs of the form $x^{\alpha+\delta} \prec x^{\beta+\delta}$.*

Proof. The proof is identical to that of Proposition 3.3.7 in Chapter 3. □

4.2 A Family with Mostly Noncoherent Term Orders

In this section we consider the family of ideals $I_{(m,n)} = \langle x^m, xy, y^n \rangle \subseteq k[x,y]$. For such an ideal $M_{(m,n)} = \{x^i : 0 \leq i \leq m-1\} \cup \{y^i : 1 \leq i \leq n-1\}$.

Any way of interleaving the sequences $x \prec x^2 \prec \ldots \prec x^{m-1}$ and $y \prec y^2 \prec \ldots \prec y^{n-1}$ is a term order, so there are $\binom{m+n-2}{n-1}$ term orders on $M_{(m,n)}$. The coherent term orders are in bijection with the 2 dimensional regions in the positive quadrant of the hyperplane arrangement $\mathcal{H}_{(m,n)}$, whose hyperplanes have normals in the set $\{(-a,b) : 0 \leq a \leq m-1, 0 \leq b \leq n-1\} \setminus (0,0)$, in the sense that two different weight vectors in the same region give rise to the same term order. The number of such regions is strictly less than mn, the number of such vectors. Thus, for fixed $n > 2$, as $m \to \infty$, we have the ratio of the number of coherent term orders to the total number of term orders is less than

$$\frac{mn}{\binom{m+n-2}{n-1}} = \frac{n!}{(m+n-2)\ldots(m+1)} \to 0$$

So the coherent term orders are only a small fraction of the total number of term orders. This answers an analogue of Question 3 from Chapter 3.

Note also that any adjacent pairs $x^a \prec y^b$ and $y^c \prec x^d$ are flippable, since they have no nontrivial multiples, so the space of term orders is connected under the flip relation.

There are, however, no noncoherent initial ideals of homogeneous ideals. This is not a deep remark; the only non-monomial homogeneous ideals are of the form $J = \langle cx^a + dy^a \rangle$, which has initial ideals $\langle x^a, y^{a+1} \rangle$ and $\langle x^{a+1}, y^a \rangle$, both of which are coherent.

Thus this family of ideals shows that there may be no noncoherent initial ideals of homogeneous ideals, even though most term orders are noncoherent.

4.3 A Noncoherent Initial Ideal

We now have the tools to provide an example of a noncoherent initial ideal.

Let S be the polynomial ring in four variables, $k[x_1, x_2, x_3, x_4]$, $A = [1\,3\,4\,7]$, and $I = \langle x^\alpha : A\alpha > 28 \rangle$. The significance of 28 is that it is the maximum degree of a binomial in the Graver basis, Gr_A, which was defined in the introduction.

Let $J = \langle x^u - x^v : Au = Av \leq 28 \rangle \subseteq S/I$. We construct a noncoherent initial ideal of J. Consider the coherent term order $\prec$ given by first comparing with the weight vector $[1,3,4,7]$, then breaking any ties with the vector $[20,25,3,0]$ and finally breaking any remaining ties with lexicographic order. The initial ideal with respect to this order is $\langle x_1^3, x_1x_2, x_2^2, x_2x_3, x_1x_4, x_1^2x_3^2, x_1x_3^4, x_2x_4^3, x_3^7\rangle$. This calculation can be done in the polynomial ring using a computer algebra package such as Macaulay 2 [20]. We now claim that $x_4^4 \prec x_3^7$ is a flippable pair of this order. It clearly has no nontrivial multiples, as x_4^4 and x_3^7 both have A-degree 28. So we need only show that the monomials are adjacent. But this follows from the fact that the order refines first by A-degree, since they are the smallest two monomials of A-degree 28 with respect to the second weight vector. We can thus create a new term order, $\prec_2$ by flipping across $x_4^4 \prec x_3^7$. We then get the initial ideal $J' = in_{\prec_2}(J) = \langle x_1^3, x_1x_2, x_2^2, x_2x_3, x_1x_4, x_1^2x_3^2, x_1x_3^4, x_2x_4^3, x_4^4\rangle$. This is the case because the given generators for J are a Gröbner basis, as S-polynomials yield either zero or another generator, and T-polynomials are either zero or a monomial multiple of that generator. So the only element of this Gröbner basis changing its initial term is $x_3^7 - x_4^4$, from which we conclude that this indeed the correct initial ideal.

Notice that, when regarded as an ideal in the polynomial ring, J' is an A-graded ideal (see Chapter 6 for more information about A-graded ideals). It is the example from Theorem 10.4 of [43] of a *noncoherent* A-graded ideal. We reproduce the proof given there to see that J' is a noncoherent initial ideal of J. Suppose that there was some coherent term order $\prec_3$ such that $J' = in_{\prec_3}(J)$. Considering the generators of J, we notice that we must have

$$\begin{aligned} x_1^2x_3 &\prec_3 x_2^2 \\ x_2x_4^2 &\prec_3 x_1x_3^4 \\ x_3^7 &\prec_3 x_4^4 \end{aligned}$$

As $\prec_3$ comes from a term order on the polynomial ring, it is valid to multiply left and right hand sides of these inequalities to obtain a new inequality. After multiplying again by the second inequality this yields $x_1^2x_2^2x_3^8x_4^4 \prec_3 x_1^2x_2^2x_3^8x_4^4$, a contradiction. We thus conclude that J' is a noncoherent initial ideal.

Chapter 5

The Vertex Ideal of a Lattice

5.1 Introduction

In this chapter we introduce and study a monomial ideal, the vertex ideal, associated to a sublattice of $\mathbb{Z}^n$. We relate algebraic properties of this ideal to combinatorial properties of the lattice. This is joint work with Serkan Hoşten.

Definition 5.1.1. Let $\mathcal{L}$ be a lattice in $\mathbb{Z}^n$ with $\dim(\mathcal{L}) = m$. For $u \in \mathbb{N}^n$ we define $P_u := \operatorname{conv}\{v \in \mathbb{N}^n : u - v \in \mathcal{L}\}$ to be the *fiber* of u with respect to $\mathcal{L}$. Clearly, if $v \in P_u$ then $P_u = P_v$.

Each fiber P_u is a rational polyhedron, by Theorem 16.1 in [39], and hence its set of vertices, $\operatorname{Vert}(P_u)$, is finite. We start with the observation (Proposition 5.2.1) that the union of all $\operatorname{Vert}(P_u)$, $u \in \mathbb{N}^n$ forms an order ideal of $\mathbb{N}^n$. We call the monomial ideal which is the complement of this order ideal the *vertex ideal* of $\mathcal{L}$, and denote it by $V_{\mathcal{L}}$.

One motivation for studying vertex ideals comes from the theory of integer programming. Suppose $A \in \mathbb{N}^{d \times n}$ is a matrix of rank d with no zero columns. Integer programming is concerned with minimizing a fixed linear form $c \cdot x$, where $c \in \mathbb{R}^n$, over $\{u \in \mathbb{N}^n : Au = b\}$ for a fixed $b \in \mathbb{N}A$. Note that if we let $\mathcal{L} = ker(A) \cap \mathbb{Z}^n$, then for $v \in \mathbb{N}^n$, the fiber of v is a polytope $\operatorname{conv}\{u \in \mathbb{N} : Au = Av\}$ (in this case we denote the fiber of v by P_b where $b = Av$). Hence studying the vertex ideal $V_{\mathcal{L}}$ in this context gives information about the vertices of all integer programming polytopes as b varies in $\mathbb{N}A$. Commutative algebra and computational algebraic geometry enter this picture through the connection between integer programming and Gröbner bases and initial ideals of the *toric* *ideal* of A (see [43], [44] and [48]).

A second motivation comes from the recent work of Saito, Sturmfels and Takayama [37] on hypergeometric differential equations. One observation these authors make is that the set of all *generic A-hypergeometric series solutions* to a GKZ A-hypergeometric system is indexed by the top-dimensional *standard pairs* of $V_{\mathcal{L}}$ where $\mathcal{L} = ker(A) \cap \mathbb{Z}^n$ (pp. 129-131 in [37]). This leads us to study the standard pairs (and hence the associated primes) of $V_{\mathcal{L}}$.

In Section 2 of this chapter we start by giving a naive algorithm to construct $V_{\mathcal{L}}$ in Theorem 5.2.3. This first algorithm needs *all* initial ideals of the associated *lattice ideal* $I_{\mathcal{L}}$, and therefore it is highly inefficient for large problems. We remedy this by giving an improved algorithm to construct a generating set for $V_{\mathcal{L}}$, using the *Graver basis* elements of $I_{\mathcal{L}}$. This second algorithm depends on a characterization of $V_{\mathcal{L}}$ which is derived from only the geometric properties of the lattice. We also describe the radical of $V_{\mathcal{L}}$ as the *Stanley-Reisner ideal* of a matroid complex.

In Section 3 the second motivation we cited above for studying $V_{\mathcal{L}}$ leads us to investigate the associated primes of $V_{\mathcal{L}}$. First we give a characterization of $Ass(V_{\mathcal{L}})$ and compute the irreducible primary decomposition of $V_{\mathcal{L}}$ in terms of a family of polytopes using similar methods to those found in [22]. This allows us to give some necessary conditions for a prime being an embedded associated prime of $V_{\mathcal{L}}$ when $\dim(\mathcal{L}) = 2$. In particular, we show that the irrelevant maximal ideal $\langle x_1, \ldots, x_n \rangle$ cannot be associated to $V_{\mathcal{L}}$ in this case. This result fails when $\dim(\mathcal{L}) \geq 3$, and we give a counterexample. This seemingly harmless counterexample turns out to be a very interesting one for our first motivation, integer programming. It provides a counterexample to a conjecture about the complexity of codimension three integer programs. More precisely, it gives a counterexample to Conjecture 6.1 in [44] which hypothesized that every cone in the *Gröbner fan* of a codimension three toric ideal has at most four facets.

In Section 4 we define another monomial ideal, $P_{\mathcal{L}}$, closely related to $V_{\mathcal{L}}$. We show that the *product ideal* $P_{\mathcal{L}}$ has the same radical as $V_{\mathcal{L}}$. In two interesting special cases, we prove that $P_{\mathcal{L}}$ (which is easier to compute) is equal to $V_{\mathcal{L}}$. The first case is when $\mathcal{L}$ comes from a unimodular matrix A. The second case is when $\mathcal{L} \subseteq \mathbb{Z}^2$ and $\dim(\mathcal{L}) = 2$. This implies that for any two dimensional lattice, we have $Top(P_{\mathcal{L}}) = Top(V_{\mathcal{L}})$.

5.2 The Minimal Generators and the Radical of the Vertex Ideal

The first goal of this section is to come up with useful characterizations of $V_{\mathcal{L}}$ which we use for devising a relatively efficient algorithm to compute it. We then give a combinatorial description of the radical of the vertex ideal. We first show the existence of the vertex ideal.

Proposition 5.2.1. *Let $\mathcal{L}$ be a lattice in $\mathbb{Z}^n$, and let P_u be a fiber of $\mathcal{L}$. For any vertex v of P_u, if $v_i > 0$, then $v - e_i$ is a vertex of P_{u-e_i} where e_i is the i-th unit vector. In other words, there exists a monomial ideal $V_{\mathcal{L}}$ in $S = k[x_1, \ldots, x_n]$ where $x^v \notin V_{\mathcal{L}}$ if and only if $v \in \mathrm{Vert}(P_u)$ for a fiber P_u of $\mathcal{L}$.*

Proof. If $v - e_i$ is not a vertex of P_{u-e_i}, then it is in the convex hull of vertices $v'_1, v'_2, \ldots, v'_k$ of P_{u-e_i}. But then v would be in the convex hull of $v'_1 + e_i, v'_2 + e_i, \ldots, v'_k + e_i$. This contradiction proves the first statement, and hence implies that the union of all $\mathrm{Vert}(P_u)$, $u \in \mathbb{N}^n$ forms an order ideal of $\mathbb{N}^n$. This is equivalent to the second statement. □

We now give a first algorithm to compute $V_{\mathcal{L}}$. To do this, we first associate a binomial ideal to $\mathcal{L}$.

Definition 5.2.2. The lattice ideal $I_{\mathcal{L}}$ is defined by

$$I_{\mathcal{L}} = \langle x^u - x^v \ : \ u, v \in \mathbb{N}^n, \ u - v \in \mathcal{L} \rangle.$$

The toric ideal defined in the introduction is a lattice ideal for $\mathcal{L} = ker(A)$. Lattice ideals have been widely studied, see for example [22], [34], [35]. In this context we are interested in the initial ideals of $I_{\mathcal{L}}$. For a *weight vector* $\omega \in \mathbb{R}^n$ such that $\omega \cdot u > 0$ for every non-zero vector $u \in \mathbb{N}^n \cap \mathcal{L}$, we let $in_\omega(I_{\mathcal{L}})$ be the ideal $\langle in_\omega(f) \ : \ f \in I_{\mathcal{L}} \rangle$ where $in_\omega(f)$ is the sum of all terms of f with maximum ω-value. If the *initial ideal* $in_\omega(I_{\mathcal{L}})$ is a monomial ideal we call ω a *generic* weight vector. Our assumption on ω ensures that each fiber P_u has a bounded face which minimizes the linear functional $u \mapsto \omega \cdot x$. Then the genericity of ω is equivalent to the condition that each such bounded face is a vertex v of P_u.

Theorem 5.2.3. *The vertex ideal $V_{\mathcal{L}}$ is equal to $\bigcap_\omega in_\omega(I_{\mathcal{L}})$ where ω is a generic weight vector.*

Proof. Since for any two lattice points $u, v \in P_u$ we have $u - v \in \mathcal{L}$, a monomial is a standard monomial of $in_\omega(I_\mathcal{L})$ if and only if its exponent vector minimizes the linear functional $u \mapsto \omega \cdot x$ over P_u [46]. Hence the monomial x^v is a standard monomial of $\bigcap_\omega in_\omega(I_\mathcal{L})$ if and only if v is the minimizer of such a functional over P_v for *some* generic weight vector. But these are precisely the vertices of the fibers of $\mathcal{L}$. □

Using this theorem we have a first algorithm for computing $V_\mathcal{L}$: compute all initial monomial ideals of $I_\mathcal{L}$ and take their intersection. We note that this is a finite algorithm, as any ideal in S has only a finite number of different initial ideals, by Corollary 2.2.1. The list of all initial ideals of $I_\mathcal{L}$ can be computed with the software `TiGERS` [24]. This first algorithm is not, however, completely satisfactory, as the number of initial ideals can be much larger than the subset needed to define the intersection. In order to illustrate this point we use the following example, where the number of initial ideals depends exponentially on the data of the lattice.

Example 5.2.4. Let $I_\mathcal{L}$ be the ideal generated by the 2×2 minors of a generic $2 \times n$ matrix $X = (x_{ij})$. This is a prime lattice ideal which is the defining ideal of the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{n-1}$ into $\mathbb{P}^{2n-1}$. Proposition 5.4 of [43] shows that with respect to the reverse lexicographic term order $x_{11} \prec \cdots \prec x_{1n} \prec x_{21} \prec \cdots \prec x_{2n}$, these 2×2 minors form a reduced Gröbner basis. By permuting columns of X, and using the corresponding reverse lexicographic term order, one gets $n!$ distinct initial ideals. This shows that $I_\mathcal{L}$ has at least $n!$ initial ideals. In Remark 5.2.13 we will see that the vertex ideal can be constructed as the intersection of only $n2^{n-1}$ initial ideals. As $\frac{n2^{n-1}}{n!} \to 0$ as $n \to \infty$, a vanishingly small proportion of the initial ideals are needed to construct $V_\mathcal{L}$ in this family.

Below we give a more efficient description of the minimal generators of the vertex ideal. For this, we need to define the Graver basis of $\mathcal{L}$.

Definition 5.2.5. Suppose $\mathcal{L} \subseteq \mathbb{Z}^n$ and let $\mathbb{R}_\rho$ be the orthant defined by the sign pattern $\rho \in \{+,-\}^n$. Then $\mathcal{L} \cap \mathbb{R}_\rho$ is a finitely generated monoid with a unique minimal generating set H_ρ, its *Hilbert basis* (Theorem 16.4 in [39]). The *Graver basis* $Gr_\mathcal{L}$ of $\mathcal{L}$ (or $I_\mathcal{L}$) is defined to be the union of all such H_ρ.

In the case that $\mathcal{L} = ker(A)$, this is the Graver basis defined in the introduction.

Lemma 5.2.6. *Suppose* $\sum_i c_i(\alpha_i - \beta_i) = 0$, *where* $c_i > 0$, *and* $\alpha_i - \beta_i \in Gr_\mathcal{L}$ *with* $\alpha_i, \beta_i \in \mathbb{N}^n$ *and* $supp(\alpha_i) \cap supp(\beta_i) = \emptyset$. *Then* $x^v = lcm_i(x^{\alpha_i})$ *is in* $V_\mathcal{L}$.

Proof. Suppose x^v is not in $V_{\mathcal{L}}$. This means that v is a vertex of P_v, so there is some $\omega \in \mathbb{R}^n$ such that $\omega \cdot v > \omega \cdot u$ for all lattice points $u \in P_v \setminus \{v\}$. But now

$$\omega \cdot v > \omega \cdot (v - (\alpha_i - \beta_i)) \text{ for each } i$$

because $\alpha_i \leq v$ means that $v' = (v - \alpha_i + \beta_i) \in \mathbb{N}^n$, and thus v' is a lattice point in $P_u \setminus \{v\}$. This implies

$$\begin{aligned}
\sum_i \omega \cdot (c_i v) &> \sum_i \omega \cdot (c_i v - c_i(\alpha_i - \beta_i)) \\
&= \sum_i \omega \cdot (c_i v) - \omega \cdot \sum_i c_i(\alpha_i - \beta_i) \\
&= \sum_i \omega \cdot (c_i v)
\end{aligned}$$

This contradiction shows that x^v is in $V_{\mathcal{L}}$. $\square$

Corollary 5.2.7. *The minimal generators of $V_{\mathcal{L}}$ are of the form as in Lemma 5.2.6.*

Proof. Let x^u be a minimal generator of $V_{\mathcal{L}}$. Hence u is not a vertex of its fiber, and so it is a convex combination $\sum_i \lambda_i v_i$ of some vertices v_i of P_u, where $0 \leq \lambda_i \leq 1$ and $\sum_i \lambda_i = 1$. Since $u - v_i$ is in $\mathcal{L}$, we have $u - v_i = \sum_j c_{ij}(\alpha_{ij} - \beta_{ij})$ where $\alpha_{ij} - \beta_{ij}$ are Graver basis elements with $\alpha_{ij} \leq u$ and $\beta_{ij} \leq v_i$, and $c_{ij} \in \mathbb{Z}_{\geq 0}$. Now clearly $\sum_i \lambda_i(u - v_i) = 0$, and thus $\sum_{i,j} \lambda_i c_{ij}(\alpha_{ij} - \beta_{ij}) = 0$. By Lemma 5.2.6, $x^v = \mathrm{lcm}_{ij}(x^{\alpha_{ij}})$ is in $V_{\mathcal{L}}$. But x^v divides x^u, and x^u is a minimal generator, so $x^u = x^v$. $\square$

Corollary 5.2.7 implies that the minimal generators of $V_{\mathcal{L}}$ can be computed by identifying all positive linear dependencies among Graver basis elements of $\mathcal{L}$. In fact only the minimal positive dependencies, known as *positive circuits*, are needed. We summarize this as follows.

Theorem 5.2.8. *Let $Gr_{\mathcal{L}} = \{\alpha_i - \beta_i\}$ be an* ordered *Graver basis of $\mathcal{L}$, so that $\alpha - \beta \in Gr_{\mathcal{L}}$ implies $\beta - \alpha \in Gr_{\mathcal{L}}$. If τ is the support of a positive circuit $\sum_{i \in \tau} c_i(\alpha_i - \beta_i) = 0$ we define x^{m_τ} to be $lcm_{j \in \tau} x^{\alpha_j}$. Then*

$$V_{\mathcal{L}} = \langle x^{m_\tau} | \tau \text{ is the support of a positive circuit of } Gr_{\mathcal{L}} \rangle.$$

Proof. If τ is the support of a positive circuit of $Gr_{\mathcal{L}}$, Lemma 5.2.6 implies that x^{m_τ} is in $V_{\mathcal{L}}$. And Corollary 5.2.7 says that every minimal generator of $V_{\mathcal{L}}$ is of this form. $\square$

Theorem 5.2.8 gives our second, more efficient, algorithm to compute $V_{\mathcal{L}}$: after computing the Graver basis $Gr_{\mathcal{L}}$, identify each positive circuit τ of $Gr_{\mathcal{L}}$ and compute $x^{m_\tau} = \mathrm{lcm}_{j\in\tau} x^{\alpha_j}$.

We observe that not all vectors of $Gr_{\mathcal{L}}$ are necessary. When $\mathcal{L} \cap \mathbb{N}^n = \{0\}$, it suffices to replace $Gr_{\mathcal{L}}$ by the ordered *universal Gröbner basis* of $\mathcal{L}$. See [43, Chapter 7] for information on computing the universal Gröbner basis.

The next result in this section describes the radical of $V_{\mathcal{L}}$. Let $B \in \mathbb{Z}^{n\times m}$ be a matrix whose columns form a basis for the m-dimensional lattice $\mathcal{L}$. We will denote the rows of B by $b_1, \ldots, b_n$. Now if ω is a generic positive cost vector, the vector ωB is contained in the relative interior of a set of m-dimensional simplicial cones with generators from $\{b_1, \ldots, b_n\}$. We call the set of all such cones C, the chamber of ω. We define Δ_ω to be the simplicial complex generated by the complementary indices of the generators of the cones in $\mathcal{C}$. By its definition, Δ_ω is an $(n-m)$-dimensional pure simplicial complex on $\{1, \ldots, n\}$. We also note that this simplicial complex is the *regular triangulation* of A with respect to ω when $\mathcal{L} = ker(A) \cap \mathbb{Z}^n$ (see Chapter 8 in [43]). Extending the connection between Stanley-Reisner ideals of regular triangulations of A and the radicals of the initial ideals of $I_{\mathcal{L}}$, we get the following proposition (Corollary 2.9 in [22], see also Section 7 in [46]). Recall that the Stanley-Reisner ideal of a simplicial complex is the ideal generated by the minimal non-faces of the complex.

Proposition 5.2.9 ([22], [46]). *The radical of $in_\omega(I_{\mathcal{L}})$ is the Stanley-Reisner ideal of the simplicial complex Δ_ω.*

Now we are ready to prove the following theorem:

Theorem 5.2.10. *The radical of $V_{\mathcal{L}}$ is $\bigcap_\sigma \langle x_i : i \in \sigma\rangle$ where the intersection is over all linearly independent subsets σ of $\{b_1, \ldots, b_n\}$ of size m, the dimension of the lattice.*

Proof.

$$\begin{aligned}
rad(V_{\mathcal{L}}) &= rad(\bigcap_{\omega\ generic} in_{\omega}(I_{\mathcal{L}})) \\
&= \bigcap_{\omega\ generic} rad(in_{\omega}(I_{\mathcal{L}})) = \bigcap_{\Delta_\omega} I_{\Delta_\omega} \\
&= \bigcap_{\Delta_\omega} \bigcap_{\tau\in\Delta_\omega} \langle x_i : i \notin \tau \rangle \\
&= \bigcap_{\sigma:\dim(\sigma)=m} \langle x_i : i \in \sigma \rangle
\end{aligned}$$

where I_{Δ_ω} is the Stanley-Reisner ideal of Δ_ω. We have the first equality on the second line because taking the radical commutes with intersections, while the second equality follows from Proposition 5.2.9. The third line is a standard result on Stanley-Reisner ideals, and the last line follows because the complement of the indices of the generators of any full dimensional simplicial cone $\{b_{i_1}, \dots, b_{i_m}\}$ is involved in some Δ_ω. □

This result can be interpreted using the notion of a matroid complex.

Definition 5.2.11. The matroid complex $\Delta(\mathcal{M})$ of a matroid $\mathcal{M}$ is the simplicial complex whose simplices are the independent sets of $\mathcal{M}$.

If $\mathcal{L} \subset \mathbb{Z}^n$ is a lattice of dimension m generated by the columns of a matrix $B \in \mathbb{Z}^{n\times m}$, then the complements of *bases* (linearly independent subsets of rows of size m) of B form the maximal independent sets of a matroid $\mathcal{M}(\mathcal{L})$. Hence the matroid complex $\Delta(\mathcal{M}(\mathcal{L}))$ is the simplicial complex whose maximal simplices are the maximal simplices occurring in Δ_ω for all generic ω. Note that when $\mathcal{L} = ker(A) \cap \mathbb{Z}^n$ for a matrix A, then $\mathcal{M}(\mathcal{L})$ is the matroid of all linearly independent subsets of the columns of A, and $\Delta(\mathcal{M}(\mathcal{L}))$ is the simplicial complex whose maximal simplices are the union of all maximal simplices appearing in all the regular triangulations of A. We now get the following corollary.

Corollary 5.2.12. *The Stanley-Reisner ideal of* $\Delta(\mathcal{M}(\mathcal{L}))$ *is the radical of* $V_{\mathcal{L}}$.

Proof. The Stanley-Reisner ideal of $\Delta(\mathcal{M}(\mathcal{L}))$ is

$$I_{\Delta(\mathcal{M}(\mathcal{L}))} = \bigcap_{\tau\in\Delta(\mathcal{M}(\mathcal{L}))} \langle x_i : i \notin \tau \rangle.$$

The above intersection can be taken over all τ where τ is a maximal face. Then since $\tau \in \Delta(\mathcal{M}(\mathcal{L}))$ if and only if $\{b_i : i \notin \tau\}$ forms a basis of B where B is matrix whose columns are a basis for $\mathcal{L}$, Theorem 5.2.10 implies that $I_{\Delta(\mathcal{M}(\mathcal{L}))} = V_{\mathcal{L}}$. ☐

Remark 5.2.13. We can now prove the last claim in Example 5.2.4. The vertex ideal $V_{\mathcal{L}}$ of the 2×2 minors of a generic $2 \times n$ matrix is a radical ideal, because the corresponding configuration is unimodular, which means all the initial ideals are radical. Hence we can use the intersection formula in the proof above. The maximal faces over which we need to take the intersection are determined by maximal independent sets of the collection $\{e_i \oplus e_j : i = 1, 2, \text{ and } 1 \leq j \leq n\}$. These are in bijection with the distinct spanning trees of the complete bipartite graph $K_{2,n}$. There are $n2^{n-1}$ such spanning trees, as exactly one vertex in the n-block is connected to both vertices in the 2-block.

Finally, we observe that the Hilbert series of $V_{\mathcal{L}}$ gives us information about the number of vertices of the fibers P_u of $\mathcal{L}$.

Proposition 5.2.14. *The Hilbert series $H_{S/V_{\mathcal{L}}}(z_1, \ldots, z_n)$ of $S/V_{\mathcal{L}}$ is $\sum_u \mathbf{z}^u$, where the sum is taken over all vertices u of all fibers P_u. When $\mathcal{L} = ker(A) \cap \mathbb{Z}^n$ for an integer matrix $A = [a_1, \ldots, a_n]$ then*

$$H_{S/V_{\mathcal{L}}}(z_1\mathbf{t}^{a_1}, \ldots, z_n\mathbf{t}^{a_n}) = \sum_{b \in \mathbb{N}A} \Big(\sum_{u \in \mathrm{Vert}(P_b)} \mathbf{z}^u \Big) \cdot \mathbf{t}^b, \ \ and,$$

$$H_{S/V_{\mathcal{L}}}(\mathbf{t}) = \sum_{b \in \mathbb{N}A} |\mathrm{Vert}(P_b)| \cdot \mathbf{t}^b.$$

We can derive information about the fibers P_u from the Hilbert function for $V_{\mathcal{L}}$. An example is given in the following proposition.

Proposition 5.2.15. *If $\mathcal{L} = ker(A) \cap \mathbb{Z}^n$ for a $1 \times n$ matrix $A = [a_1, \ldots, a_n]$ where $a_i \in \mathbb{N}$, then the number of vertices of a fiber P_u is eventually periodic, with period dividing $lcm_i(a_i)$.*

Proof. The Hilbert series $H_{S/V_{\mathcal{L}}}(t)$ can be written in the form

$$\frac{p(t)}{\prod_{i=1}^n (1 - t^{a_i})},$$

for some polynomial $p(t)$. This means that the Hilbert function of $S/V_{\mathcal{L}}$ at b, which counts the number of vertices of P_u when $Au = b$, eventually agrees with a quasi-polynomial evaluated at b. As there is an upper bound, given by the number of initial ideals of $I_{\mathcal{L}}$, on the number of vertices of any P_u, this polynomial part of the quasi-polynomial must be a constant. Since the period of the quasi-polynomial divides $\mathrm{lcm}(a_i)$, the result follows.

□

We observe that a more constructive proof of this proposition can also be given using the notion of *atomic fibers*, defined in [1].

5.3 Associated Primes and Standard Pairs of $V_{\mathcal{L}}$

With the relation between initial ideals and $V_{\mathcal{L}}$ given in Theorem 5.2.3 it is natural to ask which properties of the initial ideals of a lattice ideal pass to $V_{\mathcal{L}}$. For example, these initial ideals possess the rare property that their associated primes come in saturated chains [22]. Although we do not determine if this property holds for the vertex ideal, this section provides some tools for approaching this question. Furthermore, while investigating the associated primes of $V_{\mathcal{L}}$, we construct a lattice which provides a counterexample to a conjecture about codimension three toric ideals.

Since $V_{\mathcal{L}}$ is a monomial ideal, all of its associated primes are monomial primes of the form $\mathcal{P}_\sigma = \langle x_i : i \notin \sigma \rangle$ where $\sigma \subseteq [n] := \{1, \ldots, n\}$.

Lemma 5.3.1. *The set of associated primes* $\mathrm{Ass}(V_{\mathcal{L}})$ *of* $V_{\mathcal{L}}$ *is contained in* $\cup_\omega \mathrm{Ass}(in_\omega(I_{\mathcal{L}}))$*, the union of the associated primes of all initial ideals of* $I_{\mathcal{L}}$*. Furthermore, the set of minimal primes of* $V_{\mathcal{L}}$ *is precisely the union of the minimal primes of all initial ideals of* $I_{\mathcal{L}}$*.*

Proof. Using Theorem 5.2.3, the first statement follows from the fact that if two ideals I and J have minimal primary decompositions $\cap_i \mathcal{P}_i$ and $\cap_j \mathcal{P}'_j$, then $(\cap_i \mathcal{P}_i) \cap (\cap_j \mathcal{P}'_j)$ is a (not necessarily minimal) primary decomposition of $I \cap J$. Minimal primes of a intersection of monomials ideals are always contained in the union of the monomial primes of the ideals. The fact that this containment is an equality in this case follows from the fact, used in Theorem 5.2.10, that all minimal primes of all initial ideals have the same dimension. □

Example 5.3.2. The associated primes of $V_{\mathcal{L}}$ can be strictly contained in $\bigcup_\omega \mathrm{Ass}(in_\omega(I_{\mathcal{L}}))$. Consider the matrix $A = [1\,2\,3]$ and $\mathcal{L} = ker(A) \cap \mathbb{Z}^n$. For this lattice, the vertex ideal is $V_{\mathcal{L}} = \langle abc, a^2b, a^3c, b^3c^2 \rangle$, which has primary decomposition $\langle a^3, ab, b^3 \rangle \cap \langle a^2, ac, c^2 \rangle \cap \langle b, c \rangle$, so the associated primes of $V_{\mathcal{L}}$ are $\langle a, b \rangle$, $\langle a, c \rangle$ and $\langle b, c \rangle$. For $\omega = (100, 10, 1)$ we have $in_\omega(I_{\mathcal{L}}) = \langle a^2, ab, ac, b^3 \rangle$. This has primary decomposition $\langle a, b^3 \rangle \cap \langle a^2, b, c \rangle$, so we have $\langle a, b, c \rangle \in \cup_\omega \mathrm{Ass}(in_\omega(I_{\mathcal{L}}))$.

Corollary 3.5 of [22] gives bounds on the dimensions and codimensions of initial ideals of $I_{\mathcal{L}}$. Combined with Lemma 5.3.1 we get the following fact about the dimension and codimension of the associated primes of $V_{\mathcal{L}}$.

Proposition 5.3.3. *The dimension of an associated prime of $V_{\mathcal{L}}$ for a lattice of dimension m is at least $\max(0, n-(2^m-1))$ and the codimension is at most $\min(n, 2^m-1)$.*

For our purposes it is more convenient to study the associated primes of $V_{\mathcal{L}}$ via its *standard pairs* [45]. For a vector $u \in \mathbb{N}^n$ we denote by $\mathrm{supp}(u)$ the set $\{i : u_i \neq 0\}$.

Definition 5.3.4. An *admissible pair* of a monomial ideal M is a pair (x^u, τ) with $\tau \subseteq [n]$ such that $\mathrm{supp}(u) \cap \tau = \emptyset$, and $x^{u+v} \notin M$ for all v with $\mathrm{supp}(v) \subseteq \tau$. We place a partial order on the set of admissible pairs of M by declaring $(x^u, \tau) \prec (x^v, \sigma)$ if $x^v | x^u$ and $\mathrm{supp}(u-v) \cup \tau \subseteq \sigma$. The maximal elements of the set of admissible pairs with respect to this order are called *standard pairs.*

In the rest of the chapter we use a polyhedral characterization of the standard pairs of $V_{\mathcal{L}}$ following the results and terminology in [22] and [23]. We start with a characterization which follows from the definition of standard pairs.

Proposition 5.3.5. *The pair (x^u, τ) is a standard pair of $V_{\mathcal{L}}$ if and only if u is a vertex of P_u, $\mathrm{supp}(u) \cap \tau = \emptyset$, $u+v$ is a vertex of P_{u+v} for all v with $\mathrm{supp}(v) \subseteq \tau$, and for all $i \notin \tau$ there is some v' with support in $\tau \cup \{i\}$ such that $u+v'$ is not a vertex of $P_{u+v'}$.*

As in the previous section, let $B \in \mathbb{Z}^{n \times m}$ such that the columns of B form a lattice basis for $\mathcal{L}$. Given $u \in \mathbb{N}^n$, we can define the polyhedron $Q_u := \{x \in \mathbb{R}^m : Bx \leq u\}$. The lattice points in Q_u and the lattice points in P_u are in bijection by the correspondence $z \in Q_u \cap \mathbb{Z}^m \longleftrightarrow u - Bz \in P_u$. The origin of $\mathbb{Z}^m$ is in Q_u for all $u \in \mathbb{N}^n$ and corresponds to $u \in P_u$. We let R_u be the convex hull of the lattice points in Q_u. Note that R_u is affinely isomorphic to P_u. For a subset $\tau \subseteq [n]$ we denote by $\bar{\tau}$ the complement of τ, so $\bar{\tau} = [n] \setminus \tau$. With this convention we define $Q_u^{\bar{\tau}}$ to be the polyhedron $\{x \in \mathbb{R}^m : B^{\bar{\tau}} x \leq u^{\bar{\tau}}\}$ where the inequalities defining Q_u corresponding to τ are omitted. $R_u^{\bar{\tau}}$ denotes the convex hull of the lattice points in $Q_u^{\bar{\tau}}$. We now reformulate the characterization of standard monomials and standard pairs of $V_{\mathcal{L}}$.

Theorem 5.3.6. *The monomial x^u is a standard monomial of $V_{\mathcal{L}}$ if and only if the origin is a vertex of R_u. Moreover, a pair (x^u, τ) is a standard pair of $V_{\mathcal{L}}$ if and only if the origin is a vertex of $R_u^{\bar{\tau}}$ and it is* **not** *a vertex of $R_u^{\bar{\tau} \setminus i}$ for any $i \in \bar{\tau}$.*

Proof. The first statement follows from Theorem 5.2.3 and the fact that the origin is a vertex of R_u if and only if u is a vertex of P_u. For the second claim we use Proposition 5.3.5. The statement that u is a vertex of P_u, and $u+v$ is a vertex of P_{u+v} for all v with $\text{supp}(v) \subseteq \tau$ is equivalent to the statement that the origin is a vertex of R_u and it remains a vertex of R_{u+v} for all such v. Since $\text{supp}(v) \subseteq \tau$, this is the same thing as the origin being the vertex of $R_u^{\bar{\tau}}$. Similarly, if for all $i \notin \tau$ there exists a v' with $\text{supp}(v') \subseteq \tau \cup \{i\}$ such that the origin fails to be a vertex of $P_{u+v'}$, then the origin is also not a vertex of $R_{u+v'}$, and hence not a vertex of $R_u^{\bar{\tau}\setminus i}$, and vice versa. □

The characterization of the standard pairs in the above theorem also gives rise to a description of the irredundant irreducible primary decomposition of $V_{\mathcal{L}}$. This is very similar to the description of the irredundant irreducible primary decompositions of $in_\omega(I_{\mathcal{L}})$ given in Section 4 of [22]. In order to give this characterization we make the following definition.

Definition 5.3.7. We call the polyhedron Q_u *critical* if the origin is a vertex of R_u, but not a vertex of R_{u+e_i} for any $i = 1, \ldots, k$.

Theorem 5.3.8. *The ideal $V_{\mathcal{L}}$ has the irreducible primary decomposition*

$$V_{\mathcal{L}} = \bigcap_{Q_u^{\bar{\tau}}} \langle x_i^{u_i+1} : i \in \bar{\tau} \rangle$$

where the intersection is taken over all critical $Q_u^{\bar{\tau}}$.

Proof. The proof of Lemma 3.3 in [45] implies that

$$V_{\mathcal{L}} = \bigcap_{(x^u, \tau)} \langle x_i^{u_i+1} : i \in \bar{\tau} \rangle$$

where the intersection is taken over all standard pairs (x^u, τ) such that $x^u x_i \in V_{\mathcal{L}}$ for all $i \in \bar{\tau}$. By Theorem 5.3.6 these standard monomials are in bijection with critical $Q_u^{\bar{\tau}}$. □

When we have a two-dimensional saturated lattice $\mathcal{L} = ker(A) \cap \mathbb{Z}^n$, the codimension of $V_{\mathcal{L}}$ is two, and so Proposition 5.3.3 implies that if an embedded prime $\mathcal{P}_\tau$ of $V_{\mathcal{L}}$ exists, the codimension of $\mathcal{P}_\tau$ must be three, which means $|\tau| = n - 3$. Our next task is to show that in this case $\text{cone}\{a_i : i \in \tau\}$ cannot be a face of $\text{cone}\{a_i : i = 1, \ldots, n\}$ where a_i is the i-th column of the matrix A. The result is a consequence of the following lemma.

Lemma 5.3.9. *Let $Q \in \mathbb{R}^2$ be a polygon defined by n facet-defining inequalities $b_i \cdot x \leq u_i$, and let R be the convex hull of the lattice points in Q. Let v be a vertex of R. Then there exists a facet j of Q such that v is a vertex of the convex hull, R_j, of the lattice points in $Q_j := \{x \in \mathbb{R}^2 : b_i \cdot x \leq u_i,\ i \neq j\}$.*

Proof. Suppose not. Clearly we can assume that R is two-dimensional and that v is the origin. Let v_1 and v_2 be the two vertices of R which are the neighboring vertices of the origin, in the clockwise and counterclockwise directions respectively. We define the pointed cone K generated by v_1 and v_2, and $-K$, the opposite cone generated by $(-v_1)$ and $(-v_2)$. These constructions are illustrated in Figure 5.1. We first claim that each edge of Q has to intersect $-K$. Suppose there is an edge e, lying on the hyperplane $b_k \cdot x \leq u_k$, which does not intersect $-K$. Then the convex region $S := \{x \in \mathbb{R}^2 : b_i \cdot x \leq u_i,\ i \neq k, \text{and } b_k \cdot x \geq u_k\}$ does not intersect $-K$ as well. This is true because if $v \in S \cap -K$ there is a point w lying on e on the line segment joining v to the origin, and w would then be in $-K$. Since the origin is not a vertex of $\text{conv}(Q_k \cap \mathbb{Z}^2)$, either 0 is in the interior of an edge of R_k or it is in the interior of R_k. In the first case there exists two vertices y and z of R_k such that $y \in S$ and $z \in R \subset K$ with $0 = \lambda y + (1-\lambda)z$ for some $0 < \lambda < 1$. But then $y \in S \cap -K$, contrary to our assumption. If 0 is an interior point of R_k, then there exist three vertices y_1, y_2 and y_3 of R_k such that $0 = \lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3$ with $0 < \lambda_1, \lambda_2, \lambda_3 < 1$ and $\sum \lambda_1 + \lambda_2 + \lambda_3 = 1$. Now, either exactly one or exactly two of these vertices are in S. In the first case, say $y_1 \in S$ and $y_2, y_3 \in R$, we have $y_1 = -\frac{\lambda_2}{\lambda_1} y_2 - \frac{\lambda_3}{\lambda_1} y_3$ and hence $y_1 \in (-K) \cap S$. In the second case, say $y_1, y_2 \in S$ and $y_3 \in R$, we have $\frac{\lambda_1}{\lambda_1+\lambda_2} y_1 + \frac{\lambda_2}{\lambda_1+\lambda_2} y_2 = -\frac{\lambda_3}{\lambda_1+\lambda_2} y_3$, and hence $-\frac{\lambda_3}{\lambda_1+\lambda_2} y_3 \in (-K) \cap S$. In both cases we get a contradiction to our assumption that edge e does not intersect $-K$. This shows that all edges of Q intersect $-K$.

Because Q contains v_1 but not $-v_2$, and v_2 but not $-v_1$, some edge of Q must intersect the line segment $[v_1, -v_2]$, and another one the line segment $[v_2, -v_1]$. If we assume that the facets of Q are labeled going clockwise and the edge 1 is the first edge intersecting the facet of $-K$ defined by $(-v_1)$, then edge 1 must be the edge intersecting $[v_2, -v_1]$. And if edge n is the last edge intersecting the facet of $-K$ defined by $(-v_2)$, then edge n must be the edge intersecting $[v_1, -v_2]$. Edge 1 and edge n are the only edges of Q not lying entirely in $-K$, so they need to meet in a common vertex of Q. But their endpoints outside $-K$ are on opposite sides of the parallel line segments $[v_1, -v_2]$ and $[v_2, -v_1]$, which makes this impossible. $\square$

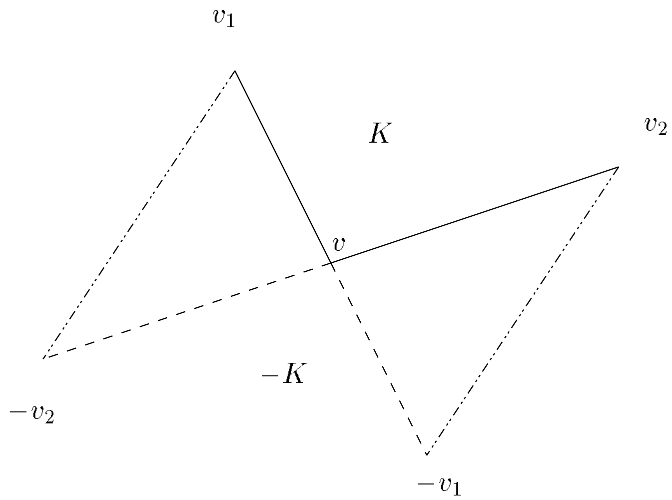

Figure 5.1: The constructions of the proof of Lemma 5.3.9

Remark 5.3.10. Note that we cannot relax the hypothesis in Lemma 5.3.9 that Q is a polygon to Q being a possibly unbounded polyhedron. An example of this phenomenon is in Figure 5.2. If any of the facets of Q are removed, the origin, O, ceases to be a vertex of R.

Theorem 5.3.11. *Let $I_{\mathcal{L}}$ be a codimension two lattice ideal where $\mathcal{L} = ker(A) \cap \mathbb{Z}^n$ with $\mathcal{L} \cap \mathbb{N}^n = \{0\}$. If $\mathcal{P}_\tau$ is an embedded prime of $V_{\mathcal{L}}$ then* $\text{cone}\{a_i : i \in \tau\}$ *is not a face of* $\text{cone}\{a_i : i = 1, \ldots, n\}$ *where a_i is the i-th column of the matrix A. In particular, the irrelevant maximal ideal $\mathcal{P}_\emptyset$ is not associated to $V_{\mathcal{L}}$.*

Proof. Let (x^u, τ) be a standard pair of $V_{\mathcal{L}}$. Suppose that $\text{cone}\{a_i : i \in \tau\}$ is a face of $\text{cone}\{a_i : i = 1, \ldots, n\}$. This means that the origin in $\mathbb{R}^2$ is in the convex hull of $\{b_i : i \in \bar{\tau}\}$, where b_i is the i-th row of the B defined after Proposition 5.3.5. This follows because positive covectors of (the oriented matroid of) A correspond to positive vectors of (the oriented matroid of) B (see [49, Chapter 6]). So $Q_u^{\bar{\tau}}$ is a polygon. Theorem 5.3.6 now implies that the origin in $\mathbb{R}^2$ is a vertex of $R_u^{\bar{\tau}}$, but not a vertex of any $R_u^{\bar{\tau}\setminus i}$ for $i \in \bar{\tau}$. But this is a contradiction to Lemma 5.3.9. □

Remark 5.3.12. The statement of Lemma 5.3.9 also cannot be generalized to higher dimensional polytopes. Similarly, the statement of Theorem 5.3.11 cannot be generalized to higher codimension. In particular, the irrelevant maximal ideal $\mathcal{P}_\emptyset$ could be an embedded prime for some $V_{\mathcal{L}}$ as the following example shows.

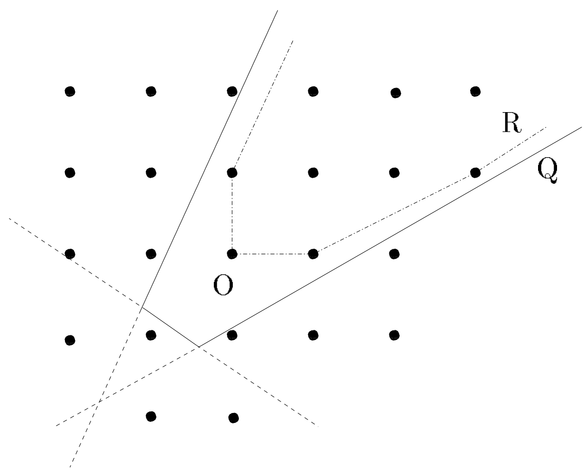

Figure 5.2: A counterexample to Lemma 5.3.9 for an unbounded polyhedron

Example 5.3.13. Let $A := [15, 247, 248, 345]$. A lattice basis for $\mathcal{L} = ker(A) \cap \mathbb{Z}^4$ is given by the columns of

$$B := \begin{bmatrix} -4 & -3 & -3 \\ -6 & 9 & -2 \\ 9 & -6 & -2 \\ -2 & -2 & 3 \end{bmatrix}.$$

If we choose $u = (9, 7, 7, 1)^T$, then $Q_u^{\{1,2,3,4\}} = Q_u = \{x \in \mathbb{R}^3 : Bx \leq u\}$ is a tetrahedron. The polytope $R_u^{\{1,2,3,4\}} = R_u$ has the following six vertices:

$$(0, 0, -3), (0, 0, 0), (1, 0, 1),$$

$$(0, 1, 1), (3, 3, 1), (23, 23, 31).$$

Now $R_u^{\{1,3,4\}}$ contains the lattice point $(-1, 0, -1)$ and the origin is in the convex hull of $\{(1, 0, 1), (-1, 0, -1)\}$; $R_u^{\{1,2,4\}}$ contains the lattice point $(0, -1, -1)$ and the origin is in the convex hull of $\{(0, 1, 1), (0, -1, -1)\}$; $R_u^{\{1,2,3\}}$ contains the lattice points $(-1, 0, 0)$ and $(0, -1, 0)$, and the origin is in the convex hull of $\{(-1, 0, 0), (0, -1, 0), (0, 0, -1), (1, 1, 1)\}$, and finally $R_u^{\{2,3,4\}}$ contains the lattice point $(-1, -1, -1)$ and the origin is in the convex hull of $\{(1, 1, 1), (-1, -1, -1)\}$. This shows the origin is not a vertex in any of these new polytopes. In particular, $(x^u, \emptyset)$ is a standard pair of $V_{\mathcal{L}}$, and hence the irrelevant ideal is an associated prime of $V_{\mathcal{L}}$.

The above example also provides a counterexample to a conjecture about the complexity of Gröbner fans of codimension three toric ideals (Conjecture 6.2 in [44]). This conjecture stated that any Gröbner cone of a codimension three toric ideal has at most four facets.

Theorem 5.3.14. *There exists a toric ideal I_A with* $\mathrm{codim}(I_A) = 3$ *which has a Gröbner cone with five facets.*

Proof. Let A be as in the above remark. If we choose $\omega = (111, 0, 342, 1)$ as the cost vector we get the following reduced Gröbner basis:

$$\{a^{23} - d, da^{10} - bc, d^{12}a^4 - b^{16}c, d^{55}a^3 - b^{76}c, d^{161}a^2 - b^{225}, d^{204}a - b^{285}, d^{247} - b^{345},$$
$$cd^9a^7 - b^{14}, cd^{20}a - b^{29}, cd^{63} - b^{89}, c^2d^8 - b^{13}a^3, c^4d^5 - b^{11}, c^5d^4 - b^{10}a^{10}, c^6d^2a^3 - b^9,$$
$$c^7a^{16} - b^8, c^7d - b^8a^7, c^8 - b^7a^{17}, bca^{13} - d^2, b^2c^2a^3 - d^3, b^3c^3 - d^4a^7, b^9a^{20} - c^6d^3,$$
$$b^{12}a^{13} - c^3d^7, b^{15}a^6 - d^{11}, b^{31}ca^2 - d^{23}, b^{44}a^5 - cd^{31}, b^{47}c^2 - d^{35}a^2, b^{60}a - d^{43}, b^{136}c - d^{98}a^2\}.$$

The corresponding Gröbner cone is given by

$$\begin{array}{rrrrcl} & +345b & & -247d & \leq & 0 \\ -20a & -9b & +6c & 3d & \leq & 0 \\ +2a & -136b & -c & +98d & \leq & 0 \\ -3a & +76b & +c & -55d & \leq & 0 \\ +7a & -3b & -3c & +4d & \leq & 0 \end{array}$$

which are all facet defining. ☐

This counterexample was found by using `TiGERS` [24], a program to compute Gröbner fans of toric ideals developed by Birkett Huber and Rekha Thomas. Computer experiments with `TiGERS` have yielded many other examples of Gröbner cones of codimension 3 toric ideals with five facets, and the following (thus far unique) codimension 3 toric ideal with a Gröbner cone with six facets.

Example 5.3.15. For the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 8 & 9 & 7 & 10 & 6 & 5 \\ 8 & 7 & 4 & 8 & 7 & 2 & 2 \\ 5 & 9 & 4 & 2 & 9 & 8 & 3 \end{bmatrix}$$

the initial ideal of I_A with respect to the weight vector $(252, 197, 0, 0, 153, 0, 0)$ corresponds to a Gröbner cone with six facets.

5.4 The Product Ideal

In this section we define the *product ideal* of $\mathcal{L}$ which is closely related to $V_{\mathcal{L}}$, and which is much easier to compute. Although in general the two ideals are not equal, we will look at two special cases where they are: the case when $\mathcal{L}$ is unimodular, and when $\mathcal{L}$ is a two-dimensional lattice in $\mathbb{Z}^2$. Even in the cases where they are not equal, we will show that the product ideal carries valuable information about $V_{\mathcal{L}}$. For instance we will show that the radicals of the two ideals are equal.

Definition 5.4.1. The product ideal $P_{\mathcal{L}}$ is the monomial ideal defined by

$$P_{\mathcal{L}} = \langle x^u x^v \,:\, u - v \in Gr_{\mathcal{L}} \rangle.$$

Since each initial ideal $in_\omega(I_{\mathcal{L}})$ contains one of x^u or x^v whenever $u - v \in \mathcal{L}$, we have $P_{\mathcal{L}} \subseteq V_{\mathcal{L}}$. This containment could be strict, however, as shown in the following example. Let $\mathcal{L} = ker(A) \cap \mathbb{Z}^3$ where $A = [3\ 4\ 5]$. Then $P_{\mathcal{L}} = \langle ab^2c, a^2bc^2, a^3bc, a^4b^3, a^5c^3, b^5c^4 \rangle$ is strictly contained in $V_{\mathcal{L}} = \langle ab^2c, a^2bc, a^4b^3, a^5c^3, b^5c^4 \rangle$. There are two special cases, though, in which the product ideal and the vertex ideal are equal. The first case is when $\mathcal{L}$ is the kernel of a unimodular matrix. We recall that a $d \times n$ matrix is unimodular if all maximal $d \times d$ minors have the same absolute value.

Proposition 5.4.2. *If $\mathcal{L} = ker(A) \cap \mathbb{Z}^n$ where A is a unimodular matrix, then $P_{\mathcal{L}} = V_{\mathcal{L}}$, and $P_{\mathcal{L}}$ coincides with the matroid ideal $I_{\Delta(\mathcal{M}(\mathcal{L}))}$.*

Proof. The initial ideals $in_\omega(I_{\mathcal{L}})$ are all squarefree (Corollary 8.9 in [43]) and hence $V_{\mathcal{L}}$ is radical. Therefore, by Corollary 5.2.12, $V_{\mathcal{L}} = I_{\Delta(\mathcal{M}(\mathcal{L}))}$. But the minimal generators of $I_{\Delta(\mathcal{M}(\mathcal{L}))}$ are of the form $\prod_{i \in J} x_i$ for some $J = \{i_1, \ldots, i_k\}$ such that $\{a_{i_1}, \ldots, a_{i_k}\}$ is a circuit of A. Now Proposition 8.11 of [43] implies that the Graver basis of $I_{\mathcal{L}}$ is $\{\alpha_i - \beta_i \in \mathcal{L} \,:\, \mathrm{supp}(\alpha_i - \beta_i)$ is the support of a circuit $\}$. Therefore $P_{\mathcal{L}} = \langle \prod_{i \in J} x_i : J = \{i_1, \ldots, i_k\}$ is the support of a circuit $\rangle$, and hence $P_{\mathcal{L}} = V_{\mathcal{L}} = I_{\Delta(\mathcal{M}(\mathcal{L}))}$. ◻

Proposition 5.4.3. *If $\mathcal{L}$ is a two-dimensional lattice in $\mathbb{Z}^2$, then $P_{\mathcal{L}} = V_{\mathcal{L}}$.*

Proof. Let $S = k[x, y]$ and suppose $x^u y^v \in V_{\mathcal{L}}$, so (u, v) is not a vertex of its fiber $P_{(u,v)}$, but $x^u y^v \notin P_{\mathcal{L}}$. If $(a, b) \in P_{(u,v)}$ where $(0, 0) \leq (a, b) \leq (u, v)$, then $x^u y^v \in P_{\mathcal{L}}$ because $x^{u-a} y^{v-b} - 1 \in I_{\mathcal{L}}$ and hence $x^{u-a} y^{v-b} \in P_{\mathcal{L}}$. So no such point in $P_{(u,v)}$ exists. Now there must be a vertex (a, b) of this fiber with $b < v$, because otherwise (u, v) would be a vertex. Let (a, b) be the vertex with $b < v$ such that (a, b) is the maximum with this property. Let H be the line through (u, v) and (a, b), let H^- be the halfspace containing the origin, and let H^+ be the other halfspace. If $P_{(u,v)} \subseteq H^+$, since (u, v) is not a vertex of $P_{(u,v)}$, the line H must contain $(c, d) \in P_{(u,v)}$ such that $0 \leq c \leq u$ and $v < d \leq 2v - b$. But then $(u - c, v - d) \in \mathcal{L}$ and $x^{u-c} - y^{d-v} \in I_{\mathcal{L}}$, which implies $x^{u-c} y^{d-v} \in P_{\mathcal{L}}$. This implies $x^u y^v \in P_{\mathcal{L}}$ since $d - v \leq v$. Hence we are reduced to the case that $P_{(u,v)}$ is not contained in H^+ and $2u < a$ (so no such $(c, d) \in P_{(u,v)}$). This means that there exists a vertex $(e, f) \in H^- \cap P_{(u,v)}$. Now if $e > u$, by the construction of (a, b) we must have $f < b$. If in addition $e < a$, the existence of a vector of the form $(k, 0) \in \mathcal{L}$ for some k means that $(e + kN, f) \in P_{(u,v)}$ for $N \gg 0$ which contradicts (a, b) being a vertex. On the other hand, if $a < e$, (a, b) would not be a vertex of $P_{(u,v)}$. So we conclude that $e < u$. But now we must have $v < f < 2v - b$, where the second inequality follows from the assumption that $(e, f) \in H^-$ and $2u < a$. Since $(e - u, f - v) \in \mathcal{L}$, it follows that $x^u y^v \in P_{\mathcal{L}}$, a contradiction, so $P_{\mathcal{L}} = V_{\mathcal{L}}$. $\square$

Example 5.4.4. Proposition 5.4.3 fails when $dim(\mathcal{L}) \geq 3$. For instance, let $\mathcal{L}$ be the lattice in $\mathbb{Z}^3$ generated by the columns of the matrix

$$\begin{bmatrix} 1 & 4 & 3 \\ -2 & 0 & 5 \\ -1 & 1 & -9 \end{bmatrix}.$$

One can verify using `Macaulay2` [20] that $P_{\mathcal{L}} = \langle ab^2c, a^4c, a^5b^2, b^8c^5, abc^{12}, b^3c^{11}, b^{19}c, ab^{21}, a^4b^{19}, ac^{26}, a^3c^{25}, b^2c^{27}, bc^{38}, a^{49}b, c^{103}, b^{103}, a^{103}\rangle$, and it is strictly contained in $V_{\mathcal{L}} = \langle c^3, ab^2c, a^4c, a^5b^2, b^{19}c, ab^{21}, a^4b^{19}, a^{49}b, b^{103}, a^{103}\rangle$.

The example at the beginning of this section shows that Proposition 5.4.3 does not hold even for a two-dimensional lattice $\mathcal{L}$ when $\mathcal{L}$ is in $\mathbb{Z}^n$ for $n \geq 3$. However, we will show that $P_{\mathcal{L}}$ and $V_{\mathcal{L}}$ have the same radical, and that for two dimensional lattices they are almost equal.

For $\sigma \subseteq \{1, \ldots, n\}$, let $\pi_\sigma : \mathbb{Z}^n \longrightarrow \mathbb{Z}^{n-|\sigma|}$ be the projection map which eliminates the coordinates indexed by σ. We will denote the image of a lattice $\mathcal{L}$ under this map by

$\mathcal{L}_\sigma$. It is clear that if $\dim(\mathcal{L}) = \dim(\mathcal{L}_\sigma)$ then $\mathcal{L}$ and $\mathcal{L}_\sigma$ are isomorphic lattices. This observation implies the following lemma.

Lemma 5.4.5. *Let $Gr_{\mathcal{L}}$ and $Gr_{\mathcal{L}_\sigma}$ be the Graver bases of the lattices $\mathcal{L}$ and $\mathcal{L}_\sigma$ where $\dim(\mathcal{L}) = \dim(\mathcal{L}_\sigma)$. Then $Gr_{\mathcal{L}_\sigma} \subseteq \pi_\sigma(Gr_{\mathcal{L}})$.*

Proof. If $\alpha' - \beta' \in Gr_{\mathcal{L}_\sigma}$, there is a unique $\alpha - \beta \in \mathcal{L}$ such that $\pi_\sigma(\alpha - \beta) = \alpha' - \beta'$. If $\alpha = \alpha_1 + \alpha_2$, $\beta = \beta_1 + \beta_2$, where $\alpha_i - \beta_i \in \mathcal{L}$ for $i = 1, 2$, and $\alpha_i, \beta_i \in \mathbb{N}^n$, then $\alpha' = \pi_\sigma(\alpha_1) + \pi_\sigma(\alpha_2)$ and $\beta' = \pi_\sigma(\beta_1) + \pi_\sigma(\beta_2)$, with $\pi_\sigma(\alpha_i) - \pi_\sigma(\beta_i) \in \mathcal{L}_\sigma$ for $i = 1, 2$. As this contradicts $\alpha' - \beta' \in Gr_{\mathcal{L}_\sigma}$, we conclude that $\alpha - \beta \in Gr_{\mathcal{L}}$, so $\alpha' - \beta' \in \pi_\sigma(Gr_{\mathcal{L}})$. □

The algebraic analogue of the projection map π_σ is the localization map $\hat{\pi}_\sigma : k[x_1, \ldots, x_n] \longrightarrow k[x_i : i \notin \sigma]$ where $\hat{\pi}_\sigma(x_i) = x_i$ if $i \notin \sigma$ and $\hat{\pi}_\sigma(x_i) = 1$ otherwise. This corresponds to localizing at the monomial prime $\mathcal{P}_\sigma = \langle x_i : i \notin \sigma \rangle$. We now compare $Top(P_{\mathcal{L}})$ with $Top(V_{\mathcal{L}})$, where $Top(M)$ is the intersection of the top-dimensional primary components of the ideal M. When we consider a monomial ideal M with top-dimensional minimal primes $\mathcal{P}_{\sigma_1}, \ldots, \mathcal{P}_{\sigma_k}$, we have $Top(M) = \cap_{i=1}^k \hat{\pi}_{\sigma_i}(M)$.

Proposition 5.4.6. *If $\dim(\mathcal{L}) = \dim(\mathcal{L}_\sigma)$ then $\hat{\pi}_\sigma(V_{\mathcal{L}}) = V_{\mathcal{L}_\sigma}$ and $\hat{\pi}_\sigma(P_{\mathcal{L}}) = P_{\mathcal{L}_\sigma}$.*

Proof. From Lemma 5.4.5 we know that $\pi_\sigma(Gr_{\mathcal{L}}) \subseteq Gr_{\mathcal{L}_\sigma}$. Let x^u be a minimal generator of $V_{\mathcal{L}}$. By Corollary 5.2.7 we know that $x^u = \mathrm{lcm}_i(x^{\alpha_i})$ where $\sum_i c_i(\alpha_i - \beta_i) = 0$ for $\alpha_i - \beta_i \in Gr_{\mathcal{L}}$ and $c_i > 0$. Now $\sum_i c_i \pi_\sigma(\alpha_i - \beta_i) = 0$. Writing $\pi_\sigma(\alpha_i - \beta_i) = \sum_j (\alpha_{ij} - \beta_{ij})$ where $\alpha_{ij} - \beta_{ij} \in Gr_{\mathcal{L}_\sigma}$ and $\alpha_{ij} \leq \alpha_i$, $\beta_{ij} \leq \beta_i$ for all j, we see that for $x^v = \mathrm{lcm}_{i,j}(x^{\alpha_{ij}})$, $x^v \in V_{\mathcal{L}_\sigma}$. Since x^v divides $\mathrm{lcm}_i(\alpha_i)$, it follows that $\hat{\pi}_\sigma(x^u) \in V_{\mathcal{L}_\sigma}$.

For the other inclusion, let x^u be a minimal generator of $V_{\mathcal{L}_\sigma}$, so $x^u = \mathrm{lcm}_i(x^{\alpha_i})$ for $\sum_i c_i(\alpha_i - \beta_i) = 0$, where $\alpha_i - \beta_i \in Gr_{\mathcal{L}_\sigma}$ and $c_i > 0$. Let $\alpha'_i - \beta'_i$ be the preimage of $\alpha_i - \beta_i$ under π_σ. We still have $\sum_i c_i(\alpha'_i - \beta'_i) = 0$, so for $x^v = \mathrm{lcm}_i(x^{\alpha'_i})$, $x^v \in V_{\mathcal{L}}$, and thus $\hat{\pi}_\sigma(x^v) = x^u \in \hat{\pi}_\sigma(V_{\mathcal{L}})$.

The second statement of the proposition follows from the definition of the product ideal, and the observation that if $\pi_\sigma(\alpha - \beta) \notin Gr_{\mathcal{L}_\sigma}$ for $\alpha - \beta \in Gr_{\mathcal{L}}$, we can write $\pi_\sigma(\alpha - \beta)$ as the sum of $\alpha_i - \beta_i \in Gr_{\mathcal{L}_\sigma}$ so that $x^{\alpha_i + \beta_i} | x^{\pi_\sigma(\alpha) + \pi_\sigma(\beta)} = \hat{\pi}_\sigma(x^{\alpha+\beta})$. □

Corollary 5.4.7. *The radical of $P_{\mathcal{L}}$ and the radical of $V_{\mathcal{L}}$ coincide. Moreover, $Top(P_{\mathcal{L}}) \subseteq Top(V_{\mathcal{L}})$.*

Proof. Theorem 5.2.10 shows that $rad(V_{\mathcal{L}})$ is an equidimensional ideal. Now an associated prime $\mathcal{P}_\sigma$ of $V_{\mathcal{L}}$ is a minimal prime if and only if $\mathcal{L}_\sigma$ is a full dimensional lattice in $\mathbb{Z}^{n-|\sigma|}$. But this is true if and only if there exist n_i such that $n_i e_i \in \mathcal{L}_\sigma$ for all $i \notin \sigma$. This happens if and only if $x_i^{n_i} \in P_{\mathcal{L}_\sigma}$ for all $i \notin \sigma$, which happens exactly whenever $P_{\mathcal{L}_\sigma} = \hat{\pi}_\sigma(P_{\mathcal{L}})$ is a zero-dimensional ideal, and hence $\mathcal{P}_\sigma$ is a minimal prime of $P_{\mathcal{L}}$. This shows that $rad(V_{\mathcal{L}}) = rad(P_{\mathcal{L}})$.

The second statement follows from Proposition 5.4.6 and the discussion before it, and the fact that $P_{\mathcal{L}_\sigma} \subseteq V_{\mathcal{L}_\sigma}$. □

Corollary 5.4.8. *If* $\dim(\mathcal{L}) = 2$ *then* $Top(P_{\mathcal{L}}) = Top(V_{\mathcal{L}})$.

Proof. Proposition 5.4.3 says that $P_{\mathcal{L}_\sigma} = V_{\mathcal{L}_\sigma}$ when $\mathcal{P}_\sigma$ is a minimal prime of $V_{\mathcal{L}}$ (and of $P_{\mathcal{L}}$). Now Proposition 5.4.6, Corollary 5.4.7 and the discussion before them imply the result. □

We note that the above corollary fails when $\dim(\mathcal{L}) \geq 3$. Example 5.4.4 provides a lattice $\mathcal{L} \in \mathbb{Z}^3$ of dimension three. Therefore $Top(V_{\mathcal{L}}) = V_{\mathcal{L}}$ and $Top(P_{\mathcal{L}}) = P_{\mathcal{L}}$, since both are Artinian ideals, but in that example we saw that $P_{\mathcal{L}} \neq V_{\mathcal{L}}$.

Part II

The toric Hilbert scheme

Chapter 6

The Toric Hilbert Scheme

6.1 Introduction

The toric Hilbert scheme is a parameter space for all ideals with the same multi-graded Hilbert function as a given toric ideal. Unlike the classical Hilbert scheme, it is unknown whether toric Hilbert schemes are connected. In this chapter we construct a graph on all the monomial ideals on the scheme, called the flip graph, and prove that the toric Hilbert scheme is connected if and only if the flip graph is connected. These graphs are used to exhibit curves in $\mathbb{P}^4$ whose associated toric Hilbert schemes have arbitrarily high dimension. We also show that the flip graph maps into the Baues graph of all triangulations of the point configuration defining the toric ideal. This is joint work with Rekha Thomas.

Let $A = [a_1 \cdots a_n]$ be a $d \times n$ integer matrix of rank d such that $ker(A) \cap \mathbb{N}^n = \{0\}$. Consider the $\mathbb{Z}^d$-graded polynomial ring $S := k[x_1, \ldots, x_n]$ over an algebraically closed field k with $\deg x_i := a_i$ for all i and an ideal $I \subseteq S$ that is homogeneous with respect to the grading by $\mathbb{N}A$, which we call *A-homogeneous*. The k-algebra $R = S/I$ is called an *A-graded algebra* if its Hilbert function $H_R(b) := \dim_k(R_b)$ is:

$$H_R(b) = \begin{cases} 1 & \text{if } b \in \mathbb{N}A \\ 0 & \text{otherwise} \end{cases}$$

The presentation ideal I is called an *A-graded ideal*, and if I is generated by monomials it is called a *monomial A-graded ideal.*

A-graded algebras were introduced by Arnold [3] who investigated matrices of the form $A = [1\ p\ q]$ where p and q are positive integers. A complete classification of all A-

graded algebras arising from one by three matrices can be found in [3], [25] and [26]. The generalization to d by n matrices is due to Sturmfels [42]. The canonical example of an A-graded ideal is the *toric ideal* of A, denoted by I_A. *Initial ideals* of I_A [43] are also A-graded.

In [42], Sturmfels constructed a parameter space whose points are in bijection with the distinct A-graded ideals in S. This variety is the underlying reduced scheme of the *toric Hilbert scheme* of A, denoted as H_A, which has been defined recently by Peeva and Stillman [32], [33]. The classical Hilbert scheme parameterizes all homogeneous, saturated ideals in S with a fixed Hilbert polynomial, where S is graded by total degree. However, unlike classical Hilbert schemes which are known to be connected [21], it is unknown whether toric Hilbert schemes are connected. Several of the techniques applied to classical Hilbert schemes cannot be used in the toric situation. In particular, the multigraded Hilbert function used to define A-graded ideals is not preserved under a change of coordinates. See [33] for further discussions. The only cases in which H_A is known to be connected are when A has corank one (i.e. $n - d = 1$) or two. In the former case the connectivity is trivial, and in the latter it follows from results in [19].

In Section 2 we define a graph on all the monomial A-graded ideals in S, called the *flip graph* of A, by defining an adjacency relation among these ideals. This generalizes the notion of adjacency between two monomial initial ideals of the toric ideal I_A, given by the edges of the *state polytope* of I_A [44]. Our main result in Section 3 reduces the connectivity of the toric Hilbert scheme to a combinatorial problem.

Theorem 6.3.1 *The toric Hilbert scheme H_A is connected if and only if the flip graph of A is connected.*

The flip graph of A provides information on the structure of H_A. In Section 4 we use these graphs to prove that two by five matrices can have toric Hilbert schemes of arbitrarily high dimension. The projective toric variety of such a matrix is a curve in $\mathbb{P}^4$.

Theorem 6.4.1 *For each $j \in \mathbb{N}\backslash\{0\}$, there exists a two by five matrix $A(j)$ such that its toric Hilbert scheme $H_{A(j)}$ has an irreducible component of dimension at least j.*

In Section 5 we relate the flip graph of A to the *Baues graph* of $\mathcal{A}$ which is a graph on all the triangulations of the point configuration $\mathcal{A} := \{a_1, \ldots, a_n\} \subset \mathbb{Z}^d$ consisting of the columns of A. The edges of the Baues graph are given by *bistellar flips*. This graph and its relatives have been studied extensively in discrete geometry [36]. Sturmfels proved that the radical of a monomial A-graded ideal I is the *Stanley-Reisner* ideal of a triangulation of $\mathcal{A}$, which we denote as $\Delta(rad(I))$ (see Theorem 4.1 in [42] or Theorem 10.10 in [43]). This gives a map from the vertices of the flip graph into the vertices of the Baues graph. We extend this map to the edges of the flip graph.

Theorem 6.5.2 *If I and I' are adjacent monomial A-graded ideals in the flip graph of A, then either they have the same radical and hence $\Delta(rad(I)) = \Delta(rad(I'))$ or $\Delta(rad(I))$ differs from $\Delta(rad(I'))$ by a bistellar flip.*

6.2 The Flip Graph of A

In this section we define an adjacency relation on all monomial A-graded ideals which, in turn, defines the flip graph of A. This graph is the main combinatorial object and tool in this paper. We first recall the definition of an A-graded ideal.

Definition 6.2.1. Let $A = [a_1 \cdots a_n] \in \mathbb{Z}^{d \times n}$ be a matrix of rank d such that $ker(A) \cap \mathbb{N}^n = \{0\}$ An ideal I in $S = k[x_1, \ldots, x_n]$ with $\deg x_i = a_i$ is called an A-graded ideal if I is A-homogeneous and $R = S/I$ has the $\mathbb{Z}^d$-graded Hilbert function:

$$H_R(b) := \dim_k(R_b) = \begin{cases} 1 & \text{if } b \in \mathbb{N}A \\ 0 & \text{otherwise} \end{cases}$$

The canonical example of an A-graded ideal is the toric ideal, which is described in the introduction. To see that I_A is A-graded, recall that $I_A = \langle x^u - x^v : Au = Av,\ u, v \in \mathbb{N}^n \rangle$, and is hence A-homogeneous. For each $b \in \mathbb{N}A$, any two monomials x^u and x^v in S of A-degree b (i.e. with $Au = Av = b$) are k-linearly dependent modulo I_A making $\dim_k((S/I_A)_b) = 1$. If $b \in \mathbb{Z}^d \backslash \mathbb{N}A$, $(I_A)_b$ is empty.

Given a *weight vector* $w \in \mathbb{N}^n$, the initial ideal of an ideal $I \subseteq S$ with respect to w is the ideal $in_w(I) := \langle in_w(f) : f \in I \rangle$ where $in_w(f)$ is the sum of all terms in f of maximal w-weight. Our assumption that $ker(A) \cap \mathbb{N}^n = \{0\}$ implies that there is a strictly positive integer vector w' in the row space of A. Using the binomial description of I_A given above, we then see that I_A is homogeneous with respect to the grading $deg(x_i) = w'_i$. Hence, the *Gröbner fan* of I_A covers $\mathbb{R}^n$ and each cell in this fan contains a non-zero non-negative integer vector in its relative interior (see Proposition 1.12 in [43]). Therefore, for any weight vector $w \in \mathbb{Z}^n$, the initial ideal $in_w(I_A)$ is well defined as it coincides with $in_{\bar{w}}(I_A)$ where $\bar{w}$ is a non-negative integer vector in the relative interior of the Gröbner cone of w. Since the Hilbert function is preserved when passing from an ideal to one of its initial ideals, all initial ideals of I_A are also A-graded.

If M is a monomial A-graded ideal, then for each $b \in \mathbb{N}A$ there is a unique monomial of degree b that does not lie in M and is hence a *standard* monomial of M. Definition 6.2.1 implies that all A-graded ideals are generated by A-homogeneous *binomials* (polynomials with at most two terms) since any two monomials of the same A-degree have to be k-linearly dependent modulo an A-graded ideal.

There is a natural action of the algebraic torus $(k^*)^n$ on S given by $\lambda \cdot x_i = \lambda_i x_i$ for $\lambda \in (k^*)^n$.

Definition 6.2.2. An A-graded ideal is said to be *coherent* if it is of the form $\lambda \cdot in_w(I_A)$ for some $\lambda \in (k^*)^n$ and $w \in \mathbb{Z}^n$.

The following lemma is a strengthening of Lemma 10.5 in [43] and was also independently discovered by Peeva and Stillman ([33, Proposition 2.2]). It uses the notion of the *Graver basis*, Gr_A, defined in the introduction. The universal Gröbner basis of an ideal is the union of all the finitely many reduced Gröbner bases of the ideal.

Lemma 6.2.3. *Let I be an A-graded ideal, and let $\mathcal{G} = \{x^{a_1} - c_1 x^{b_1}, \ldots, x^{a_k} - c_k x^{b_k}\}$ be the universal Gröbner basis of I. Here the c_i may be zero and for each binomial, $x^{a_i} - c_i x^{b_i}$, x^{a_i} and x^{b_i} are not both in I. If $c_i = 0$, choose b_i so that $Aa_i = Ab_i$ and $x^{b_i} \notin I$. Then, for all i, $x^{a_i} - x^{b_i}$ is a Graver binomial. Hence, every minimal generator of I is of this form.*

Proof. If $x^{a_i} - c_i x^{b_i} \in \mathcal{G}$, then there is some term order $\prec$ such that one of x^{a_i} and x^{b_i} is a minimal generator of $in_\prec(I)$, and the other is standard for $in_\prec(I)$. Since $in_\prec(I)$ is also A-graded, it suffices to prove the lemma for monomial A-graded ideals, where for all i we have $c_i = 0$.

Suppose there exist an i such that $x^{a_i} - x^{b_i}$ is not a Graver binomial. Then there exists $u, v \in \mathbb{N}^n$ with $Au = Av$ such that $u < a_i$ and $v < b_i$. Since I is A-graded, one of x^u or x^v is in I. If we have $x^u \in I$ then x^{a_i} would not be a minimal generator of I, and if $x^v \in I$ then x^{b_i} would not be standard. Therefore, for all i the binomial $x^{a_i} - x^{b_i}$ is a Graver binomial. $\square$

Definition 6.2.4. An A-homogeneous ideal I in S is *weakly A-graded* if $H_{S/I}(b) \in \{0,1\}$ for all $b \in \mathbb{Z}^d$. Note that we automatically have $H_{S/I}(b) = 1 \Rightarrow b \in \mathbb{N}A$.

Lemma 6.2.5. *Let I be an ideal which contains a binomial of the form $x^a - cx^b$ for every Graver binomial $x^a - x^b$. Then I is weakly A-graded.*

Proof. It suffices to prove that $M = in_{\prec}(I)$ is weakly A-graded, where $\prec$ is any term order, since $in_{\prec}(I)$ has the same Hilbert series as I. If $x^a - x^b$ is a Graver binomial, then since there is some c with $x^a - cx^b \in I$, one of x^a and x^b lies in M. Let x^u and x^v be two monomials of degree b, and let $x^a - x^b$ be a Graver binomial with $x^a | x^u$ and $x^b | x^v$. Since one of x^a and x^b lies in M, one of x^u and x^v lies in M. It thus follows that there is at most one standard monomial of M in each degree b, and so M is weakly A-graded. $\square$

We now define a "flipping" procedure on a monomial A-graded ideal which transforms this ideal into an "adjacent" monomial A-graded ideal. The idea is motivated by a similar procedure for toric initial ideals which we describe briefly.

The distinct monomial initial ideals of I_A are in bijection with the vertices of the state polytope of I_A, an $(n-d)$-dimensional polytope in $\mathbb{R}^n$ [44]. Two initial ideals are said to be adjacent if they are indexed by adjacent vertices of the state polytope. The edges of the state polytope are labeled by the binomials in the universal Gröbner basis of I_A, $UGB_A \subseteq Gr_A$.

Suppose I and I' are two adjacent monomial initial ideals of I_A connected by the edge $x^a - x^b$. The closure of the outer normal cone of the state polytope at the vertex I (respectively I') is the *Gröbner cone* K (respectively K') of I (respectively I'), the interior of which contains all the weight vectors w such that $in_w(I_A) = I$ (respectively $in_w(I_A) = I'$). The linear span of the common facet of K and K' is the hyperplane $\{u \in \mathbb{R}^n : (a-b)\cdot u = 0\}$. When w is in the interior of K, $in_w(x^a - x^b) = x^a$, x^a is a minimal generator of I and $x^b \notin I$, and when w is in the interior of K', $in_w(x^a - x^b) = x^b$, x^b is a minimal generator of I' and $x^a \notin I'$. For a w in the relative interior of the common facet of K and K', $in_w(x^a - x^b) = x^a - x^b$. Hence passing from I to I' involves "flipping" the orientation of the binomial $x^a - x^b$. No other binomial in UGB_A changes orientation during this passage. See [24] for details. We extend this notion of "flip" to all monomial A-graded ideals.

Definition 6.2.6. Let I be a monomial A-graded ideal and $x^a - x^b$ a Graver binomial with x^a a minimal generator of I and $x^b \notin I$. We define I_{flip}, the result of flipping over this binomial, to be the ideal generated by x^b, and those monomials in I other than x^a which are a term in a Graver binomial for which the other term does not lie in I. Formally, if $D_I = \{x^c : x^c - x^e \in Gr_A, x^c \in I, x^e \notin I\}$, then

$$I_{flip} := \langle x^c \in D_I : c \neq a\rangle + \langle x^b\rangle.$$

Lemma 6.2.7. *The ideal I_{flip} is weakly A-graded.*

Proof. Let $x^\alpha - x^\beta$ be a Graver binomial, with $x^\alpha \in I$. By Lemma 6.2.5 it suffices to show that either $x^\alpha \in I_{flip}$, or $x^\beta \in I_{flip}$. Since $x^\alpha \in I$, there is some (possibly identical) Graver binomial $x^{\alpha'} - x^{\beta'}$ with $x^{\alpha'}$ a minimal generator of I, and $x^{\beta'} \notin I$, and $x^{\alpha'}|x^\alpha$. If $\alpha' \neq a$, then $x^{\alpha'} \in I_{flip}$, and so $x^\alpha \in I_{flip}$. If $\alpha' = a$, then $\beta' = b$. In this case, x^a does not divide x^β because $x^\alpha - x^\beta \in Gr_A$ and hence if $x^\beta \in I$, it follows as above that $x^\beta \in I_{flip}$. Suppose $x^\beta \notin I$. If $\alpha' = a = \alpha$ then $\beta = b$ and so $x^\beta \in I_{flip}$. If $\alpha' = a \neq \alpha$ then again by definition, $x^\alpha \in I_{flip}$. ☐

As defined above, to construct I_{flip} requires knowledge of the entire Graver basis. However, the local change algorithm in [24] can be used to construct I_{flip}.

Lemma 6.2.8. *The ideal I_{flip} is the initial ideal with respect to $x^a \prec x^b$ of $W_{a-b} = \langle x^c | c \neq a, x^c$ is a minimal generator of $I\rangle + \langle x^b - x^a\rangle$.*

We note first that this initial ideal is well-defined. The only non-trivial S-pairs formed during its construction are those of a monomial with $x^b - x^a$, in which case the result is a monomial multiple of x^a, so there is never any question of what the leading term of a polynomial is. This means that I_{flip} is in fact the initial ideal of W_{a-b} with respect to *any* term order in which $x^a \prec x^b$. We call W_{a-b} a *wall ideal* because in the coherent situation, it is the initial ideal of any weight vector in the relative interior of the common facet/wall between the Gröbner cones of I and I_{flip} [24].

Proof. Let K be the initial ideal of W_{a-b} with respect to $x^a \prec x^b$. We first show the containment $K \subseteq I_{flip}$. Let x^c be a minimal generator of K. If $x^c = x^b$, or x^c is a minimal generator of I other than x^a, then $x^c \in I_{flip}$. So we need only consider the case that $c = ra + g$, where $r > 0$ and $a, b \not\leq g$, as this is the only other form minimal generators of K can have. In order to show that x^c is in I_{flip}, it suffices to show that $x^c - x^d$ is a Graver binomial, where x^d is the unique standard monomial of I of the same A-degree as x^c.

Suppose $x^c - x^d$ is not a Graver binomial, so we can write $c = \sum u_i + g'$, $d = \sum v_i + g'$, where for each i, $x^{u_i} - x^{v_i}$ is a Graver binomial. Since $x^d \notin I$, we must have $x^{u_i} \in I$ and $x^{v_i} \notin I$ for all i. If $u_i \neq a$ for some i, this would mean that x^{u_i}, and hence x^c, was in I_{flip}. We can thus reduce to the case where $g' = g$ and $v_i = b$ for all i, and so $d = rb + g$. Now since x^c is a minimal generator of K, there must be some minimal generator, x^α, of I for which the result of the reduction of the S-pair of x^α and $x^b - x^a$ is x^c. The only binomial that can be used in the reduction is $x^b - x^a$, and hence there exists $l, m \geq 0$ such that $l + m = r$ and $x^{la+mb+g}$ is the least common multiple of x^b and x^α. If $l \neq 0$, then $x^a | x^{la+mb+g} | x^{\alpha+b}$. Since x^a and x^b share no common variables, we get that $x^a | x^\alpha$ which contradicts x^α being a minimal generator of I. So we must have $l = 0$ and x^{rb+g} be the least common multiple of x^b and x^α. But this implies that $x^{rb+g} = x^d$ is a multiple of x^α and hence in I, which is a contradiction. Therefore this case cannot occur and we conclude that $K \subseteq I_{flip}$.

We now show the reverse inclusion. Suppose x^c is a minimal generator of I_{flip} not equal to x^b, and $x^c - x^d$ is the corresponding Graver binomial with $x^d \notin I$. We may assume that x^c is a multiple of x^a, as otherwise it lies in W_{a-b}, and thus in K automatically. Write $c = ra + \gamma$, where $a \not\leq \gamma$. Suppose that $x^{rb+\gamma} \notin I$. Then $d = rb + \gamma$, so we must have $\gamma = 0$ and $r = 1$ to preserve $x^c - x^d$ being a Graver binomial. But then $c = a$, contradicting x^c being a minimal generator of I_{flip}. Thus $x^{rb+\gamma} \in I$, and so there is some $\alpha \neq a$ with x^α a minimal generator of I such that $\alpha \leq rb + \gamma$. This means that $x^{rb+\gamma} \in W_{a-b}$, and so $x^{ra+\gamma} = x^c \in W_{a-b}$ because $x^b - x^a \in W_{a-b}$. Any monomial in W_{a-b} is in K, so we conclude that $x^c \in K$. This shows the other inclusion. □

Definition 6.2.9. We say that a binomial $x^a - x^b$ in the Graver basis is *flippable* for a monomial A-graded ideal I if x^a is a minimal generator of I, $x^b \notin I$ and the ideal I_{flip} obtained by flipping I over $x^a - x^b$ is again a monomial A-graded ideal.

We now give a characterization of when a binomial is flippable.

Theorem 6.2.10. *Let I be a monomial A-graded ideal, and $x^a - x^b$ a Graver binomial. Then $x^a - x^b$ is flippable for I if and only if I is the initial ideal with respect to $x^b \prec x^a$ of the wall ideal $W_{a-b} = \langle x^c | c \neq a, x^c \text{ is a minimal generator of } I\rangle + \langle x^a - x^b\rangle$.*

Proof. Since W_{a-b} is A-homogeneous, I is the initial ideal of W_{a-b} if and only if W_{a-b} is an A-graded ideal. But by Lemma 6.2.8 I_{flip} is an initial ideal of W_{a-b}, so is A-graded exactly when W_{a-b} is. $\square$

Definition 6.2.11. The *flip graph* of A has as its vertices all the monomial A-graded ideals in S. There is an edge labeled by the Graver binomial $x^a - x^b$ between two vertices I and I', if I' can be obtained from I by flipping over $x^a - x^b$.

Remark 6.2.12. The edge graph of the state polytope of I_A is a subgraph of the flip graph of A. Since the state polytope of I_A is $(n-d)$-dimensional, this subgraph is $(n-d)$-connected and so every vertex in this subgraph has valency at least $n-d$.

Let $Flips_A$ denote the set of binomials labeling the edges of the flip graph of A. Since the edges of the state polytope of I_A are labeled by the elements in UGB_A, we have $UGB_A \subseteq Flips_A \subseteq Gr_A$.

Remark 6.2.13. (i) Gasharov and Peeva [19] proved that all monomial A-graded ideals of corank two matrices are coherent. Hence, in this case, the flip graph of A is precisely the edge graph of the state polytope of I_A, which is a polygon since $n-d=2$, and $UGB_A = Flips_A$. However, even in this case, $Flips_A$ may be properly contained in Gr_A: for $A = [1\ 3\ 7]$, $UGB_A = Flips_A = \{a^2c - b^3, a^3 - b, ac^2 - b^5, b^7 - c^3, c - a^7, ab^2 - c\}$ while $Gr_A = Flips_A \cup \{a^4b - c\}$.
(ii) For $A = [1\ 3\ 4]$, $UGB_A = Flips_A = Gr_A = \{ac^2 - b^3, a^2c - b^2, b^4 - c^3, b - a^3, ab - c, a^4 - c\}$.
(iii) For $A = [3\ 4\ 5\ 13\ 14]$, $UGB_A \subsetneq Flips_A \subsetneq Gr_A$. In this case, $Flips_A \backslash UGB_A = \{a^2bcd - e^2\}$ while $Gr_A \backslash Flips_A = \{d^4 - bc^4e^2, ad^3 - bc^2e^2, e^3 - b^6cd, b^3cd^3 - e^4, e^3 - a^2c^2d^2, e^2 - ab^5c, e^3 - ab^2cd^2, e^3 - a^4bd^2\}$.

For fixed A, let S_A be the intersection of all the monomial A-graded ideals in S and let $P_A := \langle x^a x^b : x^a - x^b \in Gr_A \rangle$ be the product ideal discussed in Chapter 5. Then P_A is contained, sometimes strictly, in S_A since for each Graver binomial $x^a - x^b$, at least one of x^a or x^b belongs to each monomial A-graded ideal.

Lemma 6.2.14. *If $x^a - x^b \in Gr_A$ has at least one of x^a or x^b in P_A, then $x^a - x^b \in Gr_A \backslash Flips_A$. The converse is false.*

Proof. Suppose $x^a - x^b$ is a flippable binomial for a monomial A-graded ideal M with $x^a \in M$ and $x^b \notin M$. If $x^a \in P_A \subseteq S_A$ then $x^a \in M_{flip}$ and if $x^b \in P_A \subseteq S_A$ then $x^b \in M$

both of which are contradictions. To see that the converse is false, consider

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$

which has 29 monomial A-graded ideals all of which are coherent. The binomial $x_1x_4x_6 - x_2x_3x_5 \in Gr_A \backslash Flips_A$, but neither $x_1x_4x_6$ nor $x_2x_3x_5$ lies in $P_A = \langle x_1x_2^2x_4, x_1x_3^2x_6, x_4x_5^2x_6, x_1x_2x_3x_5, x_2x_3x_4x_5, x_2x_3x_5x_6, x_1x_2^2x_5^2x_6, x_2^2x_3^2x_4x_6, x_1x_3^2x_4x_5^2, x_1x_2x_3x_4x_5x_6 \rangle$. $\square$

6.3 Connection to the toric Hilbert Scheme

In this section we explain the relevance of flips for the toric Hilbert scheme H_A. We begin by describing the toric Hilbert scheme.

A parameter space for the set of A-graded ideals was first described by Sturmfels [42]. Peeva and Stillman improved on this construction by producing the toric Hilbert scheme of A [32], [33], which they show satisfies an important universal property. It is a version of their equations we explain below.

A degree $b \in \mathbb{N}A$ is a Graver degree if there is some Graver binomial $x^\alpha - x^\beta$ with $A\alpha = A\beta = b$. We denote by $b_1, \ldots, b_N$ the Graver degrees and by m_i the number of monomials of degree b_i. Let

$$X = \mathbb{P}^{m_1-1} \times \mathbb{P}^{m_2-1} \times \cdots \times \mathbb{P}^{m_N-1}.$$

We now describe H_A as a subscheme of X. The coordinates of each $\mathbb{P}^{m_i-1}$ can be labeled by the monomials of degree b_i as $\{\xi_u : Au = b_i\}$. A point $p \in X$ corresponds to a weakly A-graded ideal I_p by the following procedure: For each pair x^u, x^v of degree b_i, we place the binomial $\xi_v x^u - \xi_u x^v$ in I_p. For each Graver binomial $x^\alpha - x^\beta$ there thus is a binomial of the form $x^\alpha - cx^\beta$ in the resulting ideal, where c may be zero. This is immediate except in the case that $\xi_u = \xi_v = 0$. In that case, choose w with $Aw = A\alpha$ such that $\xi_w \neq 0$. Then the binomial $\xi_w x^u - \xi_u x^w \in I_p$, so $x^u \in I_p$, and so $x^u - 0 \cdot x^v$ is the required binomial. Lemma 6.2.5 now implies that I_p is weakly A-graded.

We note that the toric ideal I_A corresponds to the point in X with $\xi_u = 1$ for all u. A monomial A-graded ideal corresponds to a point in X where for each $1 \leq i \leq N$ there is exactly one $\xi_{u_i} = 1$, and $\xi_v = 0$ if $v \neq u_i$ for some i. In general, if $Au = b_i$, then $x^u \in I_p$ exactly if $\xi_u = 0$.

We now give equations for H_A, which guarantee that the resulting ideals I_p are in fact A-graded.

For each $b \in \mathbb{N}A$ we construct the matrix M_b whose d_b rows are labeled by the monomials of degree b. The n_b columns of M_b are labeled by pairs x^u, x^v of degree b such that there is some binomial $x^\alpha - x^\beta$ whose degree is a Graver degree such that $u = v - \alpha + \beta$. The corresponding column consists of ξ_α in the x^u row, $-\xi_\beta$ in the x^v row, and zeroes elsewhere.

The global equations for H_A are now given by the maximal minors of M_b for every $b \in \mathbb{N}A$. To see that these equations guarantee that I_p is A-graded, note that if I_p is not A-graded, there is some degree $b \in B$ with all monomials of degree b contained in I_p. Now homogeneous polynomials of degree b are in one-to-one correspondence with vectors in k^{d_b}. The bijection takes the basis vector with a one in the row corresponding to x^u and zeroes elsewhere to x^u, and is defined on other vectors by linear extension. Homogeneous polynomials of degree b contained in I_p are those corresponding to the image of the map $\sigma : k^{n_b} \to k^{d_b}$ given by $\sigma : x \mapsto M_b x$. Thus if all monomials of degree b are in I_p, M_b must have full rank, which means that there is a maximal minor which does not vanish.

While these equations for H_A are not binomial, it follows from the work of Peeva and Stillman [32] that each irreducible component of the scheme is given by binomial equations. The work of Eisenbud and Sturmfels on binomial ideals [11] now implies that the radical of the ideal defining each component is also a binomial ideal, and so the reduced structure on each irreducible component is a toric variety. We denote by $\widetilde{H}_A$ the underlying reduced scheme of H_A.

The main result of this section is:

Theorem 6.3.1. *The toric Hilbert scheme H_A is connected if and only if the flip graph of A is connected.*

The remainder of this section builds up to the proof of Theorem 6.3.1. In what follows we assume some familiarity with toric varieties, such as that given in [12] or [18].

Corollary 2.6 of [11] says that every prime binomial ideal determines a (not necessarily normal) toric variety. The next lemma gives a property of such varieties. When Q is a prime ideal of S we denote by $V(Q)$ the the zero set of Q in $\mathbb{A}^n$.

Lemma 6.3.2. *Consider the point configuration $\{p_1, \ldots, p_n\} \subseteq \mathbb{Z}^d$ and its toric ideal $Q = ker(\phi : k[x_1, \ldots, x_n] \to k[t^{p_1}, \ldots, t^{p_n}])$ which is a prime binomial ideal. Let v_1 and v_2 be* *two points in $V(Q) \subseteq \mathbb{A}^n$. Then v_1 and v_2 lie in the same torus orbit of $V(Q)$ if and only if they have the same support.*

Proof. The dense torus in $V(Q)$ is $V(Q) \cap (k^*)^n$, and the action of this torus on $V(Q)$ is by coordinate-wise multiplication. It thus follows that if v_1 and v_2 are in the same torus orbit, they have the same support.

Suppose $v_1, v_2 \in V(Q)$ have the same support. If this support is the entire set $\{1, \dots, n\}$, then define $u_i = (v_1)_i/(v_2)_i$. Then if $x^a - x^b$ is a binomial in Q, $u^a - u^b = \frac{v_1^a}{v_2^a} - \frac{v_1^b}{v_2^b} = \frac{1}{v_2^a v_2^b}(v_1^a v_2^b - v_1^b v_2^a) = 0$, so u is in $V(Q) \cap (k^*)^n$, and so v_1 and v_2 are in the same torus orbit.

Suppose now that v_1 and v_2 have the same support $\tau \subsetneq \{1, \dots, n\}$. Since v_1 and v_2 are in $V(Q)$, this means that there is no binomial in Q of the form $x^a - x^b$ where $\mathrm{supp}(a) \subseteq \tau$ and $\mathrm{supp}(b) \not\subseteq \tau$. This is because if such a binomial were in Q, we would have $v_i^b = 0$ for $i = 1, 2$, and $v_i^a \neq 0$ for $i = 1, 2$, which contradicts $v_1, v_2 \in V(Q)$. This means that there is no affine dependency between $\{p_i : i \in \tau\}$ and $\{p_i : i \notin \tau\}$. But this implies that $\mathrm{conv}(p_i : i \in \tau)$ is a face of $conv(p_i : 1 \leq i \leq n)$, and if $p_j \in \mathrm{conv}(p_i : i \in \tau)$, then $j \in \tau$. This means that v_1 and v_2 lie in an invariant toric subvariety, and so by a similar argument to above are torus isomorphic. □

The action of $(k^*)^n$ on A-graded ideals gives an action of $(k^*)^n$ on $\widetilde{H}_A$. The n-torus acts by mapping $v \in \widetilde{H}_A$ to $t \cdot v$ via the map $(t \cdot v)_u = t^u v_u$.We will refer to this action as the n-torus action. Two A-graded ideals ideals corresponding to points in the same n-torus orbit are said to be n-torus isomorphic. There is also a torus action on a point for every irreducible component of the reduced toric Hilbert scheme it belongs to. We will refer to these actions as the ambient torus actions. We note that these torus actions are usually different from the n-torus action, as each of the finitely many irreducible components of H_A has only finitely many ambient torus orbits, but there can be an infinite number of n-torus orbits. An example of this situation is given in Theorem 10.4 of [43]. The n-torus orbit is, however, contained inside all ambient torus orbits.

Corollary 6.3.3. *Let v be a point on $\widetilde{H}_A$. Then the n-torus orbit of v is contained in any ambient torus orbit of v.*

Proof. It is straightforward to see that $t \cdot v$ lies in every irreducible component of $\widetilde{H}_A$ in which v does (this follows from the fact that $S[l]/(lI_v + (1-l)(I_{t \cdot v}))$ is a flat $k[l]$ module).

All points in the n-torus orbit of v have the same support, and thus lie on the same ambient torus orbit by Lemma 6.3.2. $\square$

Fix an irreducible component V of $\widetilde{H}_A$. Since V is a projective toric variety, there is a polytope P corresponding to V. An ambient torus orbit of a point $v \in \widetilde{H}_A$ corresponds to a face of P. In the case of the coherent component, this polytope is the state polytope of I_A. Over the course of the next three lemmas, we show that the edges of P correspond exactly to flips.

Lemma 6.3.4. *Vertices of P correspond exactly to the monomial A-graded ideals in V.*

Proof. Let I be the ideal corresponding to a vertex p of P. The orbit of I under the ambient torus corresponding to P is just the ideal I. By Corollary 6.3.3 the n-torus orbit of I is contained in any ambient torus orbit, so I is n-torus fixed as well, and thus is a monomial ideal.

For the other implication, let I be a monomial A-graded ideal corresponding to a point v in V. As a point in X, a representation for v is as the product of standard basis vectors e_i, and thus v is invariant under any scaling of its coordinates in any fashion. It is thus is invariant under any ambient torus action, as that will be a scaling of the coordinates in some form. The point v thus corresponds to a vertex of P. $\square$

Lemma 6.3.5. *Let I be an A-graded ideal. If I has exactly two initial ideals, then I is n-torus isomorphic to an ideal of the form $J = \langle x^a - x^b, x^{c_1}, \ldots, x^{c_r}\rangle$.*

Proof. Let M_1 and M_2 be the two initial ideals of I, and let $\mathcal{G}$ be the universal Gröbner basis of I. The set $\mathcal{G}$ contains a reduced Gröbner basis for I with respect to a term order for which M_1 is the initial ideal, and so there exist binomials $x^a - cx^b \in \mathcal{G}$ with $c \neq 0$ for which x^a is a minimal generator of M_1, $x^b \notin M_1$. Suppose for all such binomials we have $x^a \in M_2$. Then $M_1 \subseteq M_2$ is an inclusion of distinct monomial A-graded ideals, which is impossible. So we conclude that there is some binomial $x^{a_1} - c_1x^{b_1} \in \mathcal{G}$ with $c_1 \neq 0$, $x^{a_1} \in M_1 \setminus M_2$ and $x^{b_1} \in M_2 \setminus M_1$.

Suppose there is some other binomial $x^{a_2} - c_2x^{b_2} \in \mathcal{G}$ with $c_2 \neq 0$. Without loss of generality we may assume that $x^{a_2} \in M_1$ and $x^{b_2} \notin M_1$. We note that $(a_1 - b_1) \neq (a_2 - b_2)$, as by Lemma 6.2.5 the two binomials $x^{a_1} - x^{b_1}$ and $x^{a_2} - x^{b_2}$ are Graver binomials, and they must be distinct since $\mathcal{G}$ is the universal Gröbner basis of I. We can thus find a supporting hyperplane for $\mathrm{pos}(a_1 - b_1, b_2 - a_2)$, which intersects the cone only at the origin. This implies the existence of a vector w which satisfies $w \cdot (a_1 - b_1) > 0$ and $w \cdot (b_2 - a_2) > 0$. Let $M = in_w(I)$. This is well defined as the Gröbner region of I is all of $\mathbb{R}^n$. Then $x^{a_1} \in M$, and $x^{b_2} \in M$, so $M \neq M_1$, and $M \neq M_2$. This means that I has a third initial ideal, which contradicts our assumption, and so we conclude that $x^{a_1} - c_1 x^{b_1}$ is the only binomial in $\mathcal{G}$.

Pick $i \in \mathrm{supp}(b_1)$. Define $\lambda_i = \frac{1}{c_1}$, and $\lambda_j = 1$ for $j \neq i$. Then λI is in the desired form. □

Theorem 6.3.6. *Let M_1 and M_2 be monomial A-graded ideals corresponding to vertices p_1 and p_2 of P. M_1 and M_2 are connected by a single flip if and only if there is an edge e of P connecting p_1 and p_2.*

Proof. Suppose p_1 and p_2 are connected by an edge e. Let I be the ideal corresponding to a point p in the relative interior of e. By Corollary 6.3.3 the n-torus closure of p is contained in e. Thus I has at most two initial ideals. If I had only one initial ideal, it would be a monomial ideal and thus corresponds to a vertex of P, by Lemma 6.3.4. We thus conclude that I has exactly two initial ideals, M_1 and M_2, corresponding to p_1 and p_2 respectively. Now by Lemma 6.3.5 I is n-torus isomorphic to $J = \langle x^a - x^b, x^{c_1}, \ldots, x^{c_r} \rangle$, where $x^a \in M_1 \setminus M_2$ and $x^b \in M_2 \setminus M_1$. Since J is A-graded, $x^a - x^b$ is a Graver binomial. Because J has initial ideals M_1 and M_2, it is their wall ideal W_{a-b}, and so M_1 and M_2 are connected by a flip over $x^a - x^b$.

Conversely, suppose M_1 and M_2 are connected by a single flip. Then there is an ideal $W_{a-b} = \langle x^a - x^b, x^{c_1}, \ldots, x^{c_r} \rangle$ which has as its two initial ideals M_1 and M_2. Let J be an A-graded ideal which is isomorphic to W_{a-b} under the ambient torus corresponding to P. Let x^d be a minimal generator of M_1, with $d \neq a$, and $x^d - x^e$ the corresponding Graver binomial with $x^e \notin M_1$. Then $x^d \in W_{a-b}$, and thus $x^d \in J$, as the ambient torus action preserves the monomials in an A-graded ideal. So J contains all minimal generators of M_1 and M_2 except x^a and x^b. Suppose J has a minimal generator $x^\alpha - c x^\beta$, where $x^\alpha - x^\beta$ is a Graver binomial, $x^\alpha, x^\beta \notin J$, and $\alpha, \beta \neq a, b$. Without loss of generality we may assume that $x^\alpha \in M_1$. If $x^\beta \notin M_1$ then $x^\alpha \in W_{a-b}$ by the definition of W_{a-b}, and thus also $x^\alpha \in J$. We thus conclude that $x^\beta \in M_1$. But this means there exist $\alpha' \leq \alpha$, $\beta' \leq \beta$, such that $x^{\alpha'}$ and $x^{\beta'}$ are minimal generators of M_1. Since x^α and x^β have disjoint support, we cannot have $\alpha' = \beta' = a$, so at least one of $x^{\alpha'}$ and $x^{\beta'}$ is in W_{a-b}. But this means at least one of x^α and x^β is in J, giving a contradiction. Hence the only binomial minimal generator of J

is of the form $x^a - c'x^b$, so as in the proof of Lemma 6.3.5 J is n-torus isomorphic to W_{a-b}. We thus see that all ambient torus closures of W_{a-b} are the same as the n-torus closure, and so p_1 and p_2 are connected by an edge. $\square$

Remark 6.3.7. The content of Lemma 6.3.4 and Theorem 6.3.6 is that the flip graph of A is the union of edge graphs of polytopes.

Proof of Theorem 6.3.1. It suffices to show that the reduced scheme $\widetilde{H}_A$ is connected if and only if the flip graph of A is connected. Since passing to an initial ideal is a flat deformation, each irreducible component contains a monomial A-graded ideal. It thus suffices to show that all monomial A-graded ideals lie in the same connected component of $\widetilde{H}_A$ if and only if the flip graph is connected. The "if" direction follows from the fact that if I_1 and I_2 are connected by a single flip, then they are both initial ideals of a single wall ideal W_{a-b}, and so lie in the same connected component of $\widetilde{H}_A$. The "only-if" direction follows from Lemmas 6.3.4 and 6.3.6, which imply that the flip graph restricted to an irreducible component of $\widetilde{H}_A$ is the edge skeleton of the polytope corresponding to that component, whose vertices are the monomial A-graded ideals in that component, and so is connected. As the intersection of two irreducible components of $\widetilde{H}_A$ contains a monomial A-graded ideal by a Gröbner deformation, this means that if $\widetilde{H}_A$ is connected, the flip graph of A is connected. $\square$

6.4 Toric Hilbert Schemes of Arbitrarily High Dimension from Curves in $\mathbb{P}^4$

In this section we exhibit toric Hilbert schemes of arbitrarily high dimensions for which the associated toric varieties are curves in $\mathbb{P}^4$. When A has corank one, its Graver basis consists of precisely one binomial $x^a - x^b$, and the flip graph of A has only the two vertices $I = \langle x^a \rangle$ and $I' = \langle x^b \rangle$ which are connected by the flip $x^a - x^b$. Hence H_A is one-dimensional and connected. All A-graded ideals of a corank two matrix are coherent [19] which implies that the flip graph of A is connected since it coincides with the edge graph of the state polytope of I_A. In this case, H_A has exactly one irreducible component which is two dimensional and smooth [33]. The toric Hilbert scheme of a corank three matrix is at least three dimensional since the irreducible component containing the coherent A-graded ideals has dimension three. In contrast to the results in coranks one and two, Theorem 6.4.1 gives a family of two by five matrices of corank three whose toric Hilbert schemes can have arbitrarily high dimensions. The projective toric variety of each matrix in the family is a curve in $\mathbb{P}^4$. Note that both the corank $n-d$ and the number of columns n are fixed for these matrices.

Theorem 6.4.1. *For each $j \in \mathbb{N}\backslash\{0\}$, the toric Hilbert scheme $H_{A(j)}$ of*

$$A(j) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3+3j & 4+3j & 6+3j \end{pmatrix}$$

has an irreducible component of dimension at least j.

These matrices were motivated by Example 5.11 in [44], and the theorem was inspired by computer experiments on their flip graphs. We first define the following monomial ideals and sets of binomials that will be used in the proof of Theorem 6.4.1. For the remainder of this section we use the polynomial ring $S = k[a,b,c,d,e]$. For each $j \in \mathbb{N}\backslash\{0\}$, let

$$P_j = \langle c^2e, bc, a^2e, ace, ae^{j+2}\rangle, \qquad R_j = \langle a^5c^j, a^8c^{j-1}, \ldots, a^{5+3(j-1)}c\rangle,$$
$$Q_j = \langle be^{j+1}, a^2c^{j+1}, b^4e^j, c^{j+2}\rangle, \quad S_j = \langle b^7e^{j-1}, b^{10}e^{j-2}, \ldots, b^{7+3(j-1)}\rangle$$

and

$$\mathcal{P}_j = \{c^2e - d^3, bc - ad, a^2e - b^2d, ace - bd^2, ae^{j+2} - c^jd^3\},$$
$$\mathcal{Q}_j = \{be^{j+1} - c^{j+1}d, a^2c^{j+1} - b^3e^j, b^4e^j - a^3c^jd, c^{j+2} - ae^{j+1}\},$$
$$\mathcal{R}_j = \{a^{5+3t}c^{j-t} - b^{6+3t}e^{(j-1)-t},\ t = 0,1,\ldots,j-1\},$$
$$\mathcal{S}_j = \{b^{7+3t}e^{(j-1)-t} - a^{6+3t}c^{(j-1)-t}d,\ t = 0,1,\ldots,j-1\}.$$

Lemma 6.4.2. *The ideal $M_j = P_j + Q_j + R_j + S_j$ is the initial ideal of $I_{A(j)}$ with respect to the weight vector $w = (1,1,2,0,2)$.*

Proof. By computing the $A(j)$-degree of both terms in each binomial of $\mathcal{G}_j := \mathcal{P}_j \cup \mathcal{Q}_j \cup \mathcal{R}_j \cup \mathcal{S}_j$, it can be seen that $\mathcal{G}_j$ is a subset of $I_{A(j)}$. It can also be checked that for each binomial in $\mathcal{G}_j$, the positive term is the leading term with respect to $w = (1,1,2,0,2)$. Hence $M_j = \langle in_w(g) : g \in \mathcal{G}_j\rangle$ is contained in the initial ideal of $I_{A(j)}$ with respect to w and no generator of M_j is redundant. The monomial ideal M_j will equal $in_w(I_{A(j)})$ if $\mathcal{G}_j$ is the reduced Gröbner basis of $I_{A(j)}$ with respect to w. Consider the elimination order $x, y \succ a,b,c,d,e$ refined by the graded reverse lexicographic order $x > y$ on the first block of variables and the weight vector w on the second block of variables. Then the reduced

Gröbner basis of $I_{A(j)}$ with respect to w is the intersection of the reduced Gröbner basis of

$$J(j) := \langle a - x, b - xy, c - xy^{3+3j}, d - xy^{4+3j}, e - xy^{6+3j}\rangle$$

with respect to $\succ$ with $k[a, b, c, d, e]$ (see Algorithm 4.5 in [43]). By a laborious check it can be shown that the reduced Gröbner basis of $J(j)$ with respect to $\succ$ is

$$\begin{aligned}
&\mathcal{G}_j \cup \{x - a, ya - b, ybd - ae, yc - d, yd^2 - ce, y^2d - e,\\
&yb^{3t+2}e^{j-t} - a^{3t+1}c^{(j+1)-t},\ t = 0, \ldots, j,\\
&y^2b^{3t+1}e^{j-t} - a^{3t}c^{(j+1)-t},\ t = 0, \ldots, j,\\
&y^{3l}b^{3t}e^{p_l-t} - a^{3t-1}c^{p_l-t+1},\ l = 1, \ldots, j,\ t = 1, \ldots, p_l := (j + 1) - l,\\
&y^{3l+1}b^{3t-1}e^{p_l-t} - a^{3t-2}c^{p_l-t+1},\ l = 1, \ldots, j,\ t = 1, \ldots, p_l,\\
&y^{3l+2}b^{3t-2}e^{p_l-t} - a^{3t-3}c^{p_l-t+1},\ l = 1, \ldots, j,\ t = 1, \ldots, p_l\}.
\end{aligned}$$

□

Lemma 6.4.3. *For each $j \in \mathbb{N}\backslash\{0\}$ the monomial $A(j)$-graded ideal M_j from Lemma 6.4.2 has exactly $2j + 4$ flippable binomials.*

Proof. We will show that the binomials in $\mathcal{Q}_j \cup \mathcal{R}_j \cup \mathcal{S}_j$ are flippable for M_j while those in $\mathcal{P}_j$ are not. In order to show that a binomial $x^a - x^b$ is flippable for M_j we need to show that every S-polynomial (monomial in our case) formed from the binomial $x^a - x^b$ (with x^a as leading term) and a minimal generator x^c of M_j different from x^a reduces to zero modulo $W_{a-b} = \langle x^c : c \neq a,\ x^c \text{ a minimal generator of } M_j\rangle + \langle x^a - x^b\rangle$.

We first consider $\mathcal{R}_j$. A binomial $a^{5+3t}c^{j-t} - b^{6+3t}e^{(j-1)-t}$ in $\mathcal{R}_j$ can form a non-trivial S-pair (S-monomial) with (i) c^2e, (ii) bc, (iii) a^2e, (iv) ace, (v) ae^{j+2}, (vi) a^2c^{j+1}, (vii) c^{j+2} and (viii) a monomial $a^{5+3l}c^{j-l}$ from R_j such that $t \neq l$. The remaining generators of M_j (except $a^{5+3t}c^{j-t}$ itself) are relatively prime to $a^{5+3t}c^{j-t}$ and so the S-pairs formed reduce to zero by Buchberger's first criterion. We consider each case separately.

(i) The S-monomials formed from c^2e and $a^{5+3t}c^{j-t} - b^{6+3t}e^{(j-1)-t}$ are $b^{6+3t}c^pe^{j-t}$, $0 \leq t \leq j - 1$, where $p = 1$ if $j - t = 1$ and $p = 0$ if $j - t > 1$.

(a) If $t = 0$, $b^6c^pe^j$ is a multiple of $b^4e^j \in Q_j$.

(b) If $t > 0$, $b^{6+3t}c^pe^{j-t}$ reduces to zero modulo $b^{7+3(t-1)}e^{j-t} \in S_j$.

(ii) The S-monomials formed from bc are $b^{7+3t}e^{(j-1)-t}$, $0 \leq t \leq j - 1$ all of which lie in S_j and hence reduce to zero modulo W_{a-b}.

(iii) The S-monomials between a^2e and $a^{5+3t}c^{j-t} - b^{6+3t}e^{(j-1)-t}$ are $b^{6+3t}e^{j-t}$ for $0 \le t \le j-1$. If $t=0$, b^6e^j is a multiple of $b^4e^j \in Q_j$, and if $t>0$ then $b^{6+3t}e^{j-t}$ is divisible by $b^{7+3(t-1)}e^{j-t} \in S_j$.

(iv) The S-monomials from ace are $b^{6+3t}e^{j-t}$ for $0 \le t \le j-1$, all of which reduce to zero as in (iii).

(v) The monomial ae^{j+2} gives $b^{6+3t}e^{2j+1-t}$ for $0 \le t \le j-1$, all of which reduce to zero modulo $be^{j+1} \in Q_j$.

(vi) From a^2c^{j+1} we get $b^{6+3t}c^{t+1}e^{(j-1)-t}$, $0 \le t \le j-1$, all of which are multiples of $bc \in P_j$.

(vii) The S-monomials from c^{j+2} are $b^{6+3t}c^{t+2}e^{(j-1)-t}$ which are also multiples of $bc \in P_j$ for $0 \le t \le j-1$.

(viii) For this last case, suppose first that $l < t \in \{0,1,2,\ldots,j-1\}$. Then $lcm(a^{5+3l}c^{j-l}, a^{5+3t}c^{j-t}) = a^{5+3t}c^{j-l}$ and the S-monomial between $a^{5+3l}c^{j-l}$ and $a^{5+3t}c^{j-t} - b^{6+3t}e^{(j-1)-t}$ is $b^{6+3t}c^{t-l}e^{(j-1)-t}$ which is a multiple of $bc \in P_j$. If $l > t$, then the S-monomial is $a^{3(l-t)}b^{6+3t}e^{(j-1)-t}$ which is divisible by $a^2e \in P_j$ since $t < l \le j-1$ and hence $t < j-1$.

Similarly, one can check that the binomials in $\mathcal{Q}_j \cup \mathcal{S}_j$ are all flippable for M_j, which shows that M_j has at least $2j+4$ flippable binomials. To finish the proof, we argue that no binomial in $\mathcal{P}_j$ is flippable for M_j.

(i) The S-binomial between $c^2e - d^3 \in \mathcal{P}_j$ and $bc \in P_j$ is bd^3 which is not divisible by any generator of M_j.

(ii) The binomials $bc - ad$, $a^2e - b^2d$ and $ace - bd^2 \in \mathcal{P}_j$ form the S-binomials ade^{j+1}, b^3de^j and $b^2d^2e^j$ respectively with $be^{j+1} \in Q_j$. None of them can be divided by a minimal generator of M_j.

(iii) The S-binomial of $ae^{j+2} - c^jd^3 \in \mathcal{Q}_j$ and $a^2e \in P_j$ is ac^jd^3 which does not lie in M_j.

Hence M_j has exactly $2j+4$ flippable binomials. □

Proof of Theorem 6.4.1. The same proof as in Lemma 6.4.3 shows that the generators of $\mathcal{I}(\mu_0,\ldots,\mu_{j-1}) := P_j + Q_j + \langle a^{5+3t}c^{j-t} - \mu_t b^{6+3t}e^{j-1-t}, t=0,\ldots,j-1\rangle + S_j$ form a Gröbner basis with respect to $w = (1,1,2,0,2)$, with initial ideal M_j, for every choice of scalars $\mu_0,\ldots,\mu_{j-1}$ from the underlying field k. Lemma 6.4.3 proved this claim for the case $\mu_i = 1$, for an $0 \le i \le j-1$ and $\mu_j = 0$ for all $j \ne i$. Since M_j is $A(j)$-graded, the $A(j)$-homogeneous ideal $\mathcal{I}(\mu_0,\ldots,\mu_{j-1})$ is also $A(j)$-graded for every choice of scalars $\mu_0,\ldots,\mu_{j-1}$. Hence there is an injective polynomial map from $\mathbb{A}_k^j \to H_{A(j)}$, such that $(\mu_0, \ldots, \mu_{j-1})$ maps to the point on $H_{A(j)}$ corresponding (uniquely) to $\mathcal{I}(\mu_0, \ldots, \mu_{j-1})$. Since $\mathbb{A}_k^j$ is irreducible, the image of this map lies entirely in one irreducible component of the toric Hilbert scheme $H_{A(j)}$ and the dimension of this component is at least $dim(\mathbb{A}_k^j) = j$. $\square$

Remark 6.4.4. In [44] it was conjectured that the maximum valency of a vertex in the state polytope of I_A is bounded above by a function in just the corank of A. As a particular case, it was also conjectured that if A is of corank three, then every vertex in the state polytope of I_A has at most four neighbors. A counterexample to this latter conjecture is found in Chapter 5, but the first is still open. Lemma 6.4.3 shows that even in corank three, a vertex in the flip graph of A can have arbitrarily many neighbors.

6.5 Connection to the Baues Problem

In this section we elaborate a connection between A-graded ideals and the Baues problem for triangulations. The Baues problem is described in the introduction, and a good reference is [36].

We relate the Baues graph to the flip graph and the toric Hilbert scheme. The connection is through the following lemma, which is a special case of Theorem 10.10 in [43]. It links monomial A-graded ideals and triangulations of $\mathcal{A}$, where A is the matrix whose columns are the points in $\mathcal{A}$, with an additional row of ones added. We will denote both the ith column of A and the ith point of $\mathcal{A}$ by a_i. We adopt the notational convention that if $\sigma \subseteq \{1, \ldots, n\}$ is a set, then $x^\sigma = \prod_{i \in \sigma} x_i$. The lemma uses *Stanley-Reisner correspondence* between simplicial complexes and squarefree monomial ideals, which is described in the introduction.

Lemma 6.5.1. *[43, Theorem 10.10] Let I be a monomial A-graded ideal. Then $\Delta(rad(I))$, the simplicial complex associated to $rad(I)$ via the Stanley-Reisner correspondence, is a triangulation of $\mathcal{A}$.*

$\square$

We can now state the main theorem of this section.

Theorem 6.5.2. *Let I be a monomial A-graded ideal and $x^a - x^b$ a flippable binomial for I. Then either $\Delta(rad(I)) = \Delta(rad(I_{flip}))$, or they differ by a bistellar flip.*

The proof will be developed through the following series of lemmas. We need to show that if I_1 and I_2 are monomial A-graded ideals which differ by a flip, then either the radicals are the same, or $\Delta(rad(I_1))$ and $\Delta(rad(I_2))$ differ by a bistellar flip. The latter involves showing:

1. $t = a - b$ is a circuit of $\mathcal{A}$ (Lemma 6.5.3).
2. C^+ is a subcomplex of $\Delta(rad(I))$ with the link of all maximal simplices of C^+ the same (Lemma 6.5.4).
3. $\Delta(rad(I_{flip}))$ differs from $\Delta(rad(I))$ exactly by replacing C^+ and its link by C^- and the corresponding link.

By a circuit of A we mean a binomial $x^a - x^b$ such that $a - b$ is a circuit of $\mathcal{A}$. Note that all circuits are Graver binomials.

Lemma 6.5.3. *Let I be a monomial A-graded ideal, with $x^a - x^b$ a flippable binomial with $x^a \in I$. Then $x^b \in rad(I) \Leftrightarrow rad(I) = rad(I_{flip})$. If $x^b \notin rad(I)$, then $x^a - x^b$ is a circuit of A.*

Proof. The implication $\Leftarrow$ is immediate in the first statement so we need only show that $x^b \in rad(I)$ implies $rad(I) = rad(I_{flip})$. Suppose $x^b \in rad(I)$. Let x^c be a minimal generator of I_{flip}. Then either x^c is a minimal generator of I, $c = b$, or $c = a + g$ for some g. In each case $x^c \in rad(I)$, so $rad(I_{flip}) \subseteq rad(I)$. If the containment is proper, Lemma 6.5.1 gives a proper containment of Stanley-Reisner ideals of triangulations of $\mathcal{A}$, which is not possible. So we conclude $rad(I) = rad(I_{flip})$.

For the second statement, suppose $x^a - x^b$ is not a circuit. Then there exists a circuit $x^c - x^d$ with $\text{supp}(c) \subseteq \text{supp}(a)$, and $\text{supp}(d) \subseteq \text{supp}(b)$ where at least one of these inclusions is proper. Since $x^b \notin rad(I)$, we must have $x^{\text{supp}(d)} \notin rad(I)$, and thus $x^d \notin I$. This implies $x^c \in I$, and so, since we know $c \neq a$, $x^c \in I_{flip}$. This means $x^{\text{supp}(c)} \in rad(I_{flip})$, and so $x^a \in rad(I_{flip})$. But this means, as above, that $rad(I_{flip}) = rad(I)$, which in turn implies that $x^b \in rad(I)$, contradicting the hypothesis. $\square$

Let I be a monomial A-graded ideal, with $x^a - x^b$ flippable, where $x^a \in I$, $x^b \notin rad(I)$. Let $t = a - b$, and $T = \text{supp}(t)$. We denote by T^+ and T^- the sets $\text{supp}(a)$ and $\text{supp}(b)$ respectively. By Lemma 6.5.3 we know that t is a circuit, so we can consider the triangulation $C^+ = \{T \setminus \{i\} : i \in T^+\}$ of C.

Lemma 6.5.4. *Let I, $x^a - x^b$, t, and C^+ be as above. Then C^+ is a subcomplex of $\Delta = \Delta(rad(I))$, and there is a subcomplex of Δ which is the common link of all maximal simplices of C^+.*

Proof. T^+ is the only minimal non-face of C^+, so to show that C^+ is a subcomplex of Δ, we need to show that x^{T^+} is the only minimal generator of $rad(I)$ with support in T.

Suppose x^c is a minimal generator of $rad(I)$, with $\mathrm{supp}(c) \subseteq T$. Then there is some $l \geq 1$ such that $x^{lc} \in I$. Write $c = a' + b'$, where $\mathrm{supp}(a') \subseteq \mathrm{supp}(a)$, and $\mathrm{supp}(b') \subseteq \mathrm{supp}(b)$. If $\mathrm{supp}(a') \neq \mathrm{supp}(a)$, then x^a does not divide x^{lc} and so x^{lc} is in the wall ideal W_{a-b}. We can choose δ with $\mathrm{supp}(\delta) \subseteq \mathrm{supp}(b)$ so that $lc + \delta = mb + a'$ for some $m \geq 1$. Since $x^{mb+a'} = x^{lc+\delta} \in W_{a-b}$, it follows that $x^{ma+a'} \in W_{a-b}$, because $x^a - x^b \in W_{a-b}$. So $x^{ma+a'} \in I_{\mathit{flip}}$, and there is thus some $p \geq m+1$ such that $x^{pa} \in I_{\mathit{flip}}$. This implies that $x^a \in rad(I_{\mathit{flip}})$. But, by Lemma 6.5.3, this means that $rad(I_{\mathit{flip}}) = rad(I)$, which in turn implies that $x^b \in rad(I)$, contradicting our hypothesis. So $\mathrm{supp}(a') = \mathrm{supp}(a) = T^+$, and thus $x^{T^+} | x^c$. This shows that x^{T^+} is the only minimal generator of $rad(I)$ with support in T, as required. From this we conclude that C^+ is a subcomplex of Δ.

We now show that every maximal simplex $\sigma \in C^+ \subseteq \Delta$ has the same link. We do this by showing that any simplex not intersecting T and not in the link of one maximal simplex of C^+ is not in the link of any other maximal simplex of C^+.

Suppose $\sigma \subseteq \{1, \ldots, n\} \setminus T$ is not a simplex in the link of a maximal simplex γ of $C^+ \subseteq \Delta$, where $\gamma = T \setminus \{p\}$ for some $p \in T^+$ and $\sigma \cap T = \emptyset$. Then $x^{\sigma \cup \gamma} \in rad(I)$, because $\sigma \cup \gamma$ is not a face of Δ, and so there exists $l \geq 1$, and x^α a minimal generator of I with $\alpha \neq a$, such that $x^\alpha | (x^{\sigma \cup \gamma})^l$. Write $\alpha = a' + b' + \sigma'$, where $\mathrm{supp}(a') \subseteq \mathrm{supp}(a)$, $\mathrm{supp}(b') \subseteq \mathrm{supp}(b)$, and $\mathrm{supp}(\sigma') \subseteq \sigma$. Choose δ with $\mathrm{supp}(\delta) \subseteq \mathrm{supp}(a)$ such that $\alpha + \delta = ma + b' + \sigma'$ for some $m \geq 0$. Then because $x^\alpha \in W_{a-b}$, we have $x^{\alpha+\delta} \in W_{a-b}$, and so $x^{mb+b'+\sigma'}$ is in W_{a-b} and thus in I. So $x^{\mathrm{supp}(b) \cup \mathrm{supp}(\sigma')} \in rad(I)$. Let τ be another maximal simplex of C^+, so $\tau = (\gamma \cup \{p\}) \setminus \{p'\}$ for some $p' \in T^+$. Then $\mathrm{supp}(b) \cup \mathrm{supp}(\sigma') \subseteq \tau \cup \sigma$, and so $x^{\tau \cup \sigma} \in rad(I)$, and thus σ is not in the link of τ in Δ. This shows that every maximal simplex $\sigma \in C^+ \subseteq \Delta$ has the same link, as required.

□

Proof of Theorem 6.5.2. If $x^b \in rad(I)$ then $rad(I) = rad(I_{\mathit{flip}})$ by Lemma 6.5.3, and so $\Delta(rad(I)) = \Delta(rad(I_{\mathit{flip}}))$.

Suppose $x^b \notin rad(I)$. Then Lemma 6.5.3 implies that $t = a - b$ is a circuit of

$\mathcal{A}$. By Lemma 6.5.4 C^+ is a subcomplex of $\Delta(rad(I))$ with each maximal simplex of C^+ having the same link in $\Delta(rad(I))$. It remains to show that $\Delta(rad(I_{flip}))$ is the result of performing a bistellar flip on $\Delta(rad(I))$.

Let Δ' be the result of performing the bistellar flip on $\Delta(rad(I))$ over t, and let M be the Stanley-Reisner ideal of Δ'.

We claim that M is the squarefree monomial ideal generated by $x^{\mathrm{supp}(b)}$, all the generators of $rad(I)$ except $x^{\mathrm{supp}(a)}$, and also all monomials of the form x^σ, such that $\mathrm{supp}(a) \subseteq \sigma$, and $\sigma \setminus (T \cap \sigma)$ is not in the link of the maximal simplices of C^+. Let $\alpha \subseteq \{1, \ldots, n\}$. Then α is a face of Δ' exactly when either α is a face of Δ and $T^- \not\subseteq \alpha$, or $\alpha = T^+ \cup \tau \cup \gamma$, where $\tau \subsetneq T^-$, and γ is in the link of the maximal simplices of C^+. This means that $\beta \subseteq \{1, \ldots, n\}$ is not a face of Δ' exactly when either $T^- \subseteq \beta$, or β is not a face of Δ and $\beta \neq T^+ \cup \tau \cup \gamma$ for any $\tau \subsetneq T^-$ and γ in the link of the maximal simplices of C^+. This proves the claim.

We now show that $rad(I_{flip}) \subseteq M$. Let x^α be a minimal generator of I_{flip} such that $x^{\mathrm{supp}(\alpha)}$ is a minimal generator of $rad(I_{flip})$. If x^α is also a minimal generator of I, then $x^{\mathrm{supp}(\alpha)}$ is in the squarefree ideal generated by all the generators of $rad(I)$ except $x^{\mathrm{supp}(a)}$, so $x^{\mathrm{supp}(\alpha)} \in M$. Since $x^{\mathrm{supp}(b)} \in M$, the only case left to consider is $\alpha = a + g$ for some $g \neq 0$ with $b \not\leq g$. Write $g = a' + b' + \gamma$, where $\mathrm{supp}(a') \subseteq \mathrm{supp}(a)$, $\mathrm{supp}(b') \subsetneq \mathrm{supp}(b)$, and $\mathrm{supp}(\gamma) \cap T = \emptyset$. Choose δ so that $\delta + a' = la + \tilde{a}$ for some $l \geq 0$, where $\mathrm{supp}(\tilde{a}) = T^+ \setminus \{p\}$ for some $p \in T^+$. Since x^α is a minimal generator of I_{flip} different from x^a, it is in W_{a-b}. It thus follows that $x^{a+g+\delta} \in W_{a-b}$, and so, because $x^a - x^b \in W_{a-b}$, we have $x^{(l+1)b+\tilde{a}+b'+\gamma} \in W_{a-b}$ and thus in I. Since $\mathrm{supp}((l+1)b + \tilde{a} + b') = T \setminus \{p\}$, $x^{(T\setminus\{p\})\cup\mathrm{supp}(\gamma)} \in rad(I)$ and thus $\mathrm{supp}(\gamma)$ is not in the link of the maximal simplices of C^+. Because $\mathrm{supp}(\gamma) = \mathrm{supp}(\alpha) \setminus T$, this means $x^\alpha \in M$, and therefore $rad(I_{flip}) \subseteq M$.

Now because $\Delta(rad(I_{flip}))$ and Δ' are both triangulations of $\mathcal{A}$, this inclusion cannot be proper. So $M = rad(I_{flip})$, and thus $\Delta(rad(I_{flip}))$ is the result of performing a bistellar flip on $\Delta(rad(I))$. □

Theorem 6.5.2 gives a map from the flip graph of monomial A-graded ideals and their flips to the graph of triangulations of $\mathcal{A}$ and their bistellar flips. Peeva [43, Theorem 10.13] has shown that this map is not always surjective. There is, however, one special case where there is an isomorphism between the two graphs. Recall that an integer matrix A is *unimodular* if all maximal minors of A have the same absolute value. A triangulation is unimodular if all simplices have volume one.

Theorem 6.5.5. *If Δ is a unimodular triangulation of A then the Stanley-Reisner ideal $I(\Delta)$ is A-graded. If A is unimodular then the flip graph of monomial A-graded ideals is isomorphic to the graph of triangulations and bistellar flips.*

Proof. The first claim is what is actually proved in Lemma 10.14 of [43], although the statement is weaker there. The second claim is an immediate consequence, as if $I(\Delta)$ is A-graded, it must be the only A-graded ideal with that radical.

□

We note that this means that a unimodular triangulation with no flips gives rise to a disconnected toric Hilbert scheme. This is true even if the whole configuration is not unimodular. The search for a disconnected toric Hilbert scheme is the topic of the next chapter.

Chapter 7

Towards a Disconnected Toric Hilbert Scheme

In this chapter we present some results which suggest the existence of a disconnected toric Hilbert scheme. These results are mostly analogous to results about the Baues Problem, which was described in the introduction.

Santos [38] has recently constructed a six dimensional point configuration with 324 points for which there is a triangulation that admits no bistellar flips. Hence this configuration has a disconnected Baues graph. By the results in [42] and the previous chapter, every monomial A-graded ideal I is supported on a triangulation of $\mathcal{A}$ via the correspondence $I \mapsto \Delta(rad(I))$, and if two monomial A-graded ideals are adjacent in the flip graph of A, then either they are supported on the same triangulation or on two triangulations that are adjacent in the Baues graph of A. Just as for monomial A-graded ideals, there is a notion of coherence for triangulations of $\mathcal{A}$. Every *coherent* triangulation of $\mathcal{A}$ (often called a *regular* triangulation in the literature) supports at least one monomial A-graded ideal, and at least one of these ideals is coherent (see Chapter 8 in [43]). On the other hand, Peeva has shown that if a triangulation of $\mathcal{A}$ is *non-coherent/non-regular* then there may be no monomial A-graded ideal whose radical is the Stanley-Reisner ideal of this triangulation (see Theorem 10.13 in [43]). Hence in order for Santos' example to lift to an example of a disconnected toric Hilbert scheme, it suffices to show that there is a monomial A-graded ideal whose radical is the Stanley-Reisner ideal of his isolated (non-regular) triangulation. A straightforward search for such a monomial A-graded ideal from his 6×324 matrix is, however, a daunting computational endeavor. Nonetheless, Santos' disconnected Baues graph seems to be evidence in favor of a disconnected flip graph.

Before Santos' isolated triangulation, one of the main pieces of evidence against the Baues graph being connected was the existence of *flip deficient* triangulations. These are triangulations of n points in $\mathbb{R}^d$ with fewer than $n-d-1$ bistellar flips. In Section 7.1 we provide analogous examples for A-graded ideals. This is again joint work with Rekha Thomas. In Section 7.2 we lay the framework for the construction of an A-graded ideal with no flips in a manner analogous to Santos' construction.

7.1 Flip Deficient A-graded Ideals

Recall that every coherent monomial A-graded ideal has at least $n-d$ neighbors in the flip graph of A, corresponding to the neighbors of its vertex in the state polytope. We say that a monomial A-graded ideal is *flip deficient* if its valency in the flip graph of A is strictly less than $n-d$. All flip deficient monomial A-graded ideals are necessarily non-coherent. Before Santos constructed an isolated triangulation, discrete geometers provided several examples of flip deficient triangulations (triangulations with valency less that $n-d$ in the Baues graph) as evidence in support of the existence of a disconnected Baues graph. We provide examples of flip deficient monomial A-graded ideals.

Theorem 7.1.1. *For each matrix $A(n) := [1\ 2\ 3\ 7\ 8\ 9\ a_7 \cdots a_n]$ with $a_i \in \mathbb{N}$ and $9 < a_7 < \cdots < a_n$, there is a monomial $A(n)$-graded ideal with at most $n-3 < n-1 = corank(A(n))$ flips.*

Proof. For the matrix $A = [1\ 2\ 3\ 7\ 8\ 9]$, the monomial ideal $J = \langle\, x_1x_5, x_2x_4, x_1x_4, x_1x_2, x_4x_6, x_2x_6, x_1x_6, x_3x_4, x_2^2x_3, x_1x_3, x_2x_5^2, x_2^2x_5, x_1^2, x_3^2, x_2^4, x_3x_5^3, x_4^2x_5^2, x_4^3, x_5^6, x_4x_5^4\,\rangle$ is A-graded. The flippable binomials of J are $x_5^6 - x_3x_6^5$, $x_2x_6 - x_3x_5$ and $x_3^2 - x_2^3$. In this example, there are 2910 monomial A-graded ideals in total and the flip graph of A is connected.

Consider the monomial ideal $J' = J + \langle x_7, \ldots, x_n\rangle \subseteq k[x_1, \ldots, x_n]$ and a degree $b \in \mathbb{N}A(n) = \mathbb{N}A = \mathbb{N}$. All the monomials in $k[x_1, \ldots, x_n]$ of $A(n)$-degree b that are divisible by at least one of $x_7, \ldots, x_n$ are in J' by construction. Among the monomials in $k[x_1, \ldots, x_6]$ of degree b (there is at least one such since $b \in \mathbb{N}A$), there is precisely one that is not in J and hence not in J' and hence J' is $A(n)$-graded. If $x^a - x^b \in k[x_1, \ldots, x_6]$ is flippable for

J' then

$$in_{x^a \succ x^b}(\langle x^a - x^b \rangle + \langle x^c : x^c \text{ minimal generator of } J,\, c \neq a \rangle + \langle x_7, \ldots, x_n \rangle) = J'.$$

The only non-trivial S-pairs that are produced during this calculation are those between $x^a - x^b$ and a monomial minimal generator x^c of J. Since the resulting initial ideal equals J', it follows that $in_{x^a \succ x^b}(\langle x^a - x^b \rangle + \langle x^c : x^c \text{ minimal generator of } J,\, c \neq a \rangle) = J$ and hence $x^a - x^b$ is flippable for J. So $x^a - x^b$ must be one of the three flippable binomials of J. Additionally, each of the minimal generators $x_7, \ldots, x_n$ of J' provides a flippable binomial and hence J' has $3 + (n - 6) = n - 3$ flippable binomials. □

Remark 7.1.2. We have not found matrices of corank three with flip deficiency in our experiments. However, flip deficiency occurs in corank four. Consider $A = [3\ 6\ 8\ 10\ 15]$ and its monomial A-graded ideal

$$\langle ae, bd, ab^2, be, a^2, d^2, e^2, b^3, abc^2 \rangle.$$

The neighboring monomial A-graded ideals are:
$\langle ae, bd, ab^2, be, de, a^2, d^2, e^2, b^3 \rangle$ from $de - abc^2$,
$\langle ae, bd, ab^2, b^2e, a^2, d^2, e^2, b^3, acd, abc^2 \rangle$ from $acd - be$, and
$\langle ae, bd, ab^2, ad^2, be, b^2c, a^2, d^3, e^2, b^3, abc^2, d^2e \rangle$ from $b^2c - d^2$.

Remark 7.1.3. The above computations were made using two different programs. Starting with a monomial initial ideal of the toric ideal I_A one can compute all monomial A-graded ideals in the same connected component as this initial ideal by using the results in Section 2 of Chapter 6 to calculate all the neighbors of a monomial A-graded ideal. This computation can be done using the program `TiGERS` [24] with the command `tigers -iAe filename` where `filename` is the standard input file for `TiGERS` with the data of the matrix A. In order to find all monomial A-graded ideals, we resort to a second program that first computes the Graver basis of A and then systematically constructs weakly A-graded monomial ideals by choosing one monomial from each Graver binomial to be in the ideal (cf. Lemma 6.2.5). The program then compares the Hilbert series of each such ideal against that of an initial ideal of I_A to decide if it is A-graded. Comparing the total number of ideals produced by the two programs gives a convenient way to decide if the flip graph is connected.

We conclude this section with an algorithmic issue concerning the enumeration of all A-graded monomial ideals in the same connected component as a fixed one. The main program in `TiGERS` enumerates the vertices of the state polytope of I_A by using the *reverse search* strategy of Avis and Fukuda [4], which requires only the current vertex to be stored at any given time. The input to the program is any one monomial initial ideal of I_A from which the program reconstructs all the others without needing to consult the list of ideals it has already found. An essential requirement of this algorithm is a method by which the input ideal can be distinguished from any other monomial initial ideal of I_A by considering only the edges of the state polytope. This is done in `TiGERS` as follows:

Suppose M_1 and M_2 are two monomial initial ideals of I_A induced by the weight vectors w_1 and w_2 respectively. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be the corresponding reduced Gröbner bases of I_A. Then for each facet binomial $x^a - x^b$ in $\mathcal{G}_1$ we have $w_1 \cdot (a - b) > 0$ and for each facet binomial $x^\alpha - x^\beta \in \mathcal{G}_2$ we have $w_2 \cdot (\alpha - \beta) > 0$. The reduced Gröbner bases $\mathcal{G}_1$ and $\mathcal{G}_2$ coincide if and only if each facet binomial $x^\alpha - x^\beta$ of $\mathcal{G}_2$ satisfies the inequality $w_1 \cdot (\alpha - \beta) > 0$. Suppose the input is a fixed initial ideal of I_A. By the previous observation, every other monomial initial ideal of I_A will have a mismarked facet binomial with respect to this term order and hence can be distinguished from the input ideal. The following example shows that monomial A-graded ideals cannot be distinguished by checking the orientation of their flippable binomials.

Example 7.1.4. Consider $A = [3\ 4\ 5\ 13\ 14]$ and its non-coherent monomial A-graded ideal

$$M = \langle cd^5, c^2e^3, be, d^9, b^2, c^3, a^6, bd, ae^2, ad^3, ac^2, a^2d, a^2b, bc, a^3e, a^3c \rangle.$$

The flippable binomials of M are $ae^2 - cd^2$, $c^3 - a^5$ and $d^9 - ce^8$. With respect to the weight vector $w = (0, 0, 1, 20, 22)$, each of these flippable binomials has its positive term as leading term and hence M cannot be distinguished from $in_w(I_A)$ by checking whether its flippable binomials are mismarked with respect to w.

This means that we cannot use a reverse search strategy to compute all monomial A-graded ideals in the same connected as the toric ideal, but must instead keep and consult a list of previously found ideals. This is memory intensive, and limits the size of examples which can be computed.

7.2 A Method to Construct an Isolated A-graded Ideal

In this section we provide a method to construct an isolated A-graded ideal, given a monomial A'-graded ideal with certain, hopefully less restrictive, properties, where A' is related to A. This method is based on Santos' construction of an isolated triangulation in [38]. Many of the lemmas are direct analogues of Santos' lemmas. The proofs, however, are often trickier, and in some cases we are able to prove only more restrictive (but sufficient) versions of his lemmas. The difficulty in these cases arises from the difference in behavior between the set of circuits and the Graver basis of A.

Let A_i be a $d_i \times n_i$ integer matrix for $i = 1, 2$. Throughout this section M_1 is a monomial A_1 graded ideal in the ring $R_1 = k[x_1, \ldots, x_{n_1}]$, and M_2 is a monomial A_2 graded ideal in the ring $R_2 = k[y_1, \ldots, y_{n_2}]$, with I_{A_1} and I_{A_2} homogeneous ideals. By $A_1 \times A_2$ we mean the matrix whose columns are the elements of the Cartesian product of the columns of A_1 and A_2. More precisely, $A_1 \times A_2$ is a $(d_1 + d_2) \times (n_1 n_2)$ matrix whose columns consist of a column of A_1 in the first d_1 coordinates, and then a column of A_2 in the next d_2 coordinates.

Definition 7.2.1. Let $R = k[x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2}]$, and $S = k[z_{ij}, 1 \leq i \leq n_1, 1 \leq j \leq n_2]$. Viewing M_1 as an ideal in the x_i and M_2 as an ideal in the y_i, define a ring map $\phi : S \to R/(M_1 + M_2)$ by $z_{ij} \mapsto x_i y_j$. We denote the kernel of this map by $M_1 \times M_2$. We also note two other ring maps from S to R_1 and R_2. These are the projection maps $\pi_1 : z_{ij} \mapsto x_i$ and $\pi_2 : z_{ij} \mapsto y_j$. We will also use π_i to denote the similar projection from $\mathbb{Z}^{d_1 \times d_2}$ to $\mathbb{Z}^{d_i}$ sending e_{kl} to e_k, if $i = 1$, or e_l, if $i = 2$, where e_{kl} and e_k are standard basis vectors for $\mathbb{Z}^{d_1 \times d_2}$ and $\mathbb{Z}^{d_1}$.

We note that a monomial z^u does not lie in $M_1 \times M_2$ if and only if $\pi_1(z^u) \notin M_1$ and $\pi_2(z^u) \notin M_2$.

Lemma 7.2.2. *The ideal $M_1 \times M_2$ is an $A_1 \times A_2$ graded ideal.*

Proof. In every degree $b = (b_1, b_2)$ there is at least one monomial z^u such that $x^{\pi_1(u)} = x^{s_1} \notin M_1$ and $y^{\pi_2(u)} = y^{s_2} \notin M_2$. However every standard monomial of degree b maps to $x^{s_1} y^{s_2}$ under the action of ϕ, so for every pair of standard monomials z^{u_1}, z^{u_2} of degree b there is a binomial $z^{u_1} - z^{u_2} \in M_1 \times M_2$. Thus we have $dim_k((S/(M_1 \times M_2)_b) = 1$, as required. ∎

The map ϕ factors through the Segre embedding, so the binomial minimal generators of $M_1 \times M_2$ are those coming from the ideal defining the Segre product.

Lemma 7.2.3. *Every binomial minimal generator of $M_1 \times M_2$ is of the form $z^u - z^v$ where $\pi_i(u) = \pi_i(v)$ for $i = 1, 2$, and $\pi_i(u)_j \in \{0, 1\}$.*

Proof. Since $M_1 \times M_2$ is an $A_1 \times A_2$ graded ideal, if $z^u - z^v$ is a binomial minimal generator, then it is a Graver basis element for $A_1 \times A_2$ (by Lemma 6.2.3), and neither z^u nor z^v lie in $M_1 \times M_2$. Thus we must have $\pi_i(u) = \pi_i(v)$ for $i = 1, 2$.

So it suffices to show that all Graver basis elements $z^u - z^v$ for $A_1 \times A_2$ satisfying $\pi_i(u - v) = 0$ for $i = 1, 2$ have the desired form. Given a Graver basis element $z^u - z^v$ with the required projections, we construct the following bipartite directed weighted graph. There are n_1 vertices of the first type, labeled $x_1, \ldots, x_{n_1}$, and n_2 vertices of the second type, labeled $y_1, \ldots, y_{n_2}$. If $u_{ij} > 0$, we place a directed edge from x_i to y_j of weight u_{ij}. If $v_{ij} > 0$, we place a directed edge from y_i to x_j of weight v_{ij}. Consider the underlying directed graph, with the weights erased. Since $\pi_i(u - v) = 0$, each vertex with an arc entering it has at least one leaving it, and vice versa. This means the graph has no sources or sinks, and so is not acyclic. Take a minimal cycle, which visits each vertex at most once, and construct the binomial corresponding to placing a weight of one on every edge. This binomial, $z^{u'} - z^{v'}$, satisfies $u' \leq u$, $v' \leq v$, and $z^{u'} - z^{v'} \in M_1 \times M_2$. We thus conclude that $u = u'$ and $v = v'$, so $\pi_i(u)_j \in \{0, 1\}$ for $1 \leq j \leq n_i$ and $i = 1, 2$.

□

We will call such Graver basis elements *Segre Graver binomials.*

Definition 7.2.4. Let Δ be an abstract simplicial complex. A *locally acyclic* orientation on the 1-skeleton of Δ is an orientation of all the edges of Δ such that the orientation restricted to any simplex of Δ is acyclic. Such an orientation need not be globally acyclic. An edge in such an orientation is *reversible* if switching its orientation, and leaving all other orientations constant, results in another locally acyclic orientation of Δ.

We now describe a particular monomial refinement of $M_1 \times M_2$.

Definition 7.2.5. Let Δ_i be the abstract simplicial complex consisting of those subsets of $[n_i]$ which are the support of some standard monomial of M_i. Fix a locally acyclic orientation on the 1-skeleton of Δ_i. This gives an ordering on the vertices in each simplex of Δ_i. If σ is a simplex of Δ_1 and τ is a simplex of Δ_2, let $\prec_{\sigma,\tau}$ be the term order on $S_{\sigma,\tau} = k[z_{ij} : i \in \mathrm{supp}(\sigma), j \in \mathrm{supp}(\tau)]$ which is reverse lexicographic with respect to ordering of the variables of $S_{\sigma,\tau}$ with $z_{ij} \prec z_{kl}$ if $i < k$ or if $i = k$ and $j < l$. We now define an ideal $M = \mathrm{Stair}(M_1 \times M_2)$ which is a refinement of $M_1 \times M_2$. For each Graver basis element of $A_1 \times A_2$ of the form $z^u - z^v$, where $\pi_1(u) = \pi_1(v) = \sigma$ and $\pi_2(u) = \pi_2(v) = \tau$, place $in_{\prec_{\sigma,\tau}}(z^u - z^v)$ in M. Finally, add to M any monomial $z^u \in M_1 \times M_2$.

Lemma 7.2.6. *The monomial ideal M described above is an $A_1 \times A_2$ graded ideal.*

Proof. Since a choice has been made for every Graver basis element, M is at least weakly $A_1 \times A_2$ graded. Suppose there are no standard monomials of M of degree $b = (b_1, b_2)$. Let x^u be the unique standard monomial of M_1 of degree b_1, and y^v be the unique standard monomial of M_2 of degree b_2. Let $\sigma = \mathrm{supp}(u)$, and $\tau = \mathrm{supp}(v)$. There is at least one monomial z^w of degree b which maps to $x^u y^v$. By our assumption all such monomials are in M, so for each such z^w there is a Segre Graver basis element of $A_1 \times A_2$, $z^\alpha - z^\beta$, with $z^\alpha \in M$ and $\alpha \leq w$. Such Graver basis elements satisfy $z^\alpha \succ_{\sigma,\tau} z^\beta$. In particular there exist such α and β for z^a, where z^a is the least monomial with respect to $\prec_{\sigma,\tau}$ of degree b which satisfies $\pi_1(z^a) = x^u$ and $\pi_2(z^a) = y^v$. But now $z^{a-\alpha+\beta} \prec_{\sigma,\tau} z^a$, and $z^{a-\alpha+\beta}$ has the required projections, contradicting the minimality of z^a. This shows that M is $A_1 \times A_2$ graded.

□

Lemma 7.2.7. *If z^u is a minimal generator of $M = \mathrm{Stair}(M_1 \times M_2)$ with $z^u - z^v$ a Segre Graver binomial such that $z^v \notin M$, then $z^u = z_{ij} z_{kl}$ for some i, j, k, l with $i < k$ and $l < j$ in the order given by the orientations of the 1-skeleton of Δ_1 and Δ_2.*

Proof. By the definition of the staircase refinement, we know that for all quadratic Segre Graver binomials of the form $z_{ij}z_{kl} - z_{il}z_{kj}$ with $i < k$ and $l < j$ we have $z_{ij}z_{kl} \in M$. So it remains to prove that these are the only minimal generators of M coming from Segre Graver binomials. Suppose z^u is a minimal generator of M coming from a Segre Graver binomial $z^u - z^v$ of degree greater than 2. Since $\mathrm{supp}(\pi_1(u)) \in \Delta_1$ and $\mathrm{supp}(\pi_2(u)) \in \Delta_2$, the orientations on the respective one skeletons give an ordering of the vertices of $\mathrm{supp}(\pi_1(u))$ and $\mathrm{supp}(\pi_2(u))$. After reordering we may assume that this ordering is consistent with the naming of the x and y variables. Now since z^u is a minimal generator of M, whenever $z_{ij}z_{kl} | z^u$ with $i < k$ we must have $j < l$. Let z_{mn} be the greatest variable with respect to $\prec_{\sigma,\tau}$ dividing z^u or z^v. Since $\pi_i(u - v) = 0$ for $i = 1, 2$, there are variables z_{mp} and z_{qn} dividing the other of z^u and z^v with $p < n$ and $q < m$. But as above we know that $z_{mp}z_{qn} \in M$, contradicting either z^u being a minimal generator or z^v being standard for M.

We thus conclude that every minimal generator of M arising from a Segre Graver binomial is quadratic. $\square$

We aim to show that if M_1 and M_2 satisfy certain conditions, then M has no flips. The first condition we place on M_2 is that it should be squarefree. It is possible that this condition may be unnecessary in the following lemma, which is a partial analogue of Santos' Lemma 2.1.

Lemma 7.2.8. *Let $M = \mathrm{Stair}(M_1 \times M_2)$, where M_2 is a squarefree ideal, $z^u \notin M$, and γ be such that $x^{\pi_1(u)+\gamma} \notin M_1$. Then there exists a $\tilde{\gamma}$ such that $z^{u+\tilde{\gamma}} \notin M$ and $\pi_1(\tilde{\gamma}) = \gamma$.*

Proof. It suffices to prove this for x^γ a variable, so we henceforth assume $\gamma = x_i$. Since $x^{\pi_1(u)+\gamma} \notin M_1$, $\mathrm{supp}(\pi_1(u)+\gamma) \in \Delta_1$, and so the orientation of the one skeleton of Δ_1 gives an ordering of the vertices of $\mathrm{supp}(\pi_1(u)+\gamma)$. After relabeling we can assume that this ordering on $\mathrm{supp}(\pi_1(u)+\gamma)$ agrees with the ordering of the x variables. Similarly we may assume that the ordering of the y variables agrees with the ordering on $\mathrm{supp}(\pi_2(u))$ given by the orientation on Δ_2.

Let $k = \max_j\{j \in \mathrm{supp}(\pi_1(u)) \text{ and } j \leq i\}$. Let $l = \max_j\{e_{kj} \leq u\}$. Choose $\tilde{\gamma}$ to be e_{il}. Then $\pi_1(\tilde{\gamma}) = \gamma$, and $\mathrm{supp}(\pi_2(u+\tilde{\gamma})) = \mathrm{supp}(\pi_2(u))$. Since M_2 is squarefree, it thus follows that $y^{\pi_2(u+\tilde{\gamma})} \notin M_2$. Since we already know $x^{\pi_1(u+\tilde{\gamma})} \notin M_1$, it follows that if $z^{u+\tilde{\gamma}} \in M$, then it must be divisible by some z^α where $z^\alpha - z^\beta$ is a Segre Graver binomial and $z^\alpha \succ z^\beta$ induced from the ordering on $\pi_1(u) \cup \{i\}$ and $\pi_2(u)$. By Lemma 7.2.7, we must have $z^\alpha = z_{ab}z_{cd}$, with $a < c$ and $d < b$. Since $z^u \notin M$, it must be the least element with respect to the staircase term order with the same projections in its degree, so it follows that if z_{ab} and z_{cd} both divide z^u, with $a < c$, then $b < d$. Thus one of witnesses z_{ab} or z_{cd} must equal z_{il}. But k and l have been chosen so as not to be part of such a pair. This means that such a z^α cannot exist, and so $z^{u+\tilde{\gamma}} \notin M$. $\square$

The following is a partial analogue of Santos' Lemma 2.3 [38].

Lemma 7.2.9. *Let $M = \mathrm{Stair}(M_1 \times M_2)$ where M_2 is a squarefree ideal. Let $z^u - z^v$ be a flippable binomial for M such that $\pi_1(u) \neq \pi_1(v)$. Then the following hold:*

1. *If $u_{ij} > 0$ for some i and j, then there is no j' such that $v_{ij'} > 0$.*

2. *$x^{\pi_1(u)} - x^{\pi_1(v)}$ is a flippable binomial for M_1.*

If in addition the total degree of z^u is not equal to the total degree of any minimal generator of M_2, then

3. $\pi_2(u) = \pi_2(v)$.

Proof. In order for $z^u - z^v$ to be a flippable binomial, a necessary condition is that z^u be a minimal generator of M, and $z^v \notin M$. Since $\pi_1(u) \neq \pi_1(v)$, $x^{\pi_1(u)}$ must be a minimal generator of M_1. This follows because M_1 is A_1-graded, so at least one of $x^{\pi_1(u)}$ and $x^{\pi_1(v)}$ must be in M_1. Since $z^v \notin M$, we must have $x^{\pi_1(u)} \in M_1$. If it were not a minimal generator of M_1, then z^u would not be a minimal generator of M. Now since $x^{\pi_1(u)}$ is a minimal generator of M_1, and $x^{\pi_1(v)} \notin M_1$ is of the same degree, $x^{\pi_1(u)} - x^{\pi_1(v)}$ must be a Graver basis element for M_1, so $\mathrm{supp}(\pi_1(u)) \cap \mathrm{supp}(\pi_1(v)) = \emptyset$. This proves the first part of the lemma.

We now show that $x^{\pi_1(u)} - x^{\pi_1(v)}$ is a flippable binomial for M_1. Let x^w be another minimal generator of M_1. We need to show that $S(x^w, x^{\pi_1(u)} - x^{\pi_1(v)})$ is in M_1. Suppose not. Let $x^\gamma = x^w/\gcd(x^w, x^{\pi_1(u)})$. Then we are assuming that $x^{\pi_1(v)+\gamma} \notin M_1$. Since $x^v \notin M$, Lemma 7.2.8 gives the existence of a $\tilde{\gamma}$ such that $z^{v+\tilde{\gamma}} \notin M$. Choose a w' such that $\pi_1(w') = w - \gamma$ and $z^{w'} | z^u$. This is possible as $x^{w-\gamma} = \gcd(x^w, x^{\pi_1(u)})$. Set $\tilde{w} = w' + \tilde{\gamma}$. As $\pi_1(\tilde{w}) = w$, $z^{\tilde{w}} \in M$, so since $z^u - z^v$ is flippable we must have $S(z^{\tilde{w}}, z^u - z^v) \in M$. But this divides $z^{v+\tilde{\gamma}}$, which we know is not in M. This is a contradiction, so it follows that for all minimal generator x^w of M_1, $S(x^w, x^{\pi_1(u)} - x^{\pi_1(v)})$ is in M_1, and thus that $x^{\pi_1(u)} - x^{\pi_1(v)}$ is a flippable binomial for M_1.

Suppose that M_2 does not have a minimal generator of degree $deg(z^u) = \sum_i u_i$, but that $\pi_2(u) \neq \pi_2(v)$. Since M_2 is A_2-graded, we would then have $y^{\pi_2(u)} \in M_2$, as we know $y^{\pi_2(v)} \notin M_2$. If $y^{\pi_2(u)}$ is not a minimal generator of M_2, then z^u would not be a minimal generator of M, so $y^{\pi_2(u)}$ is a minimal generator of M_2. But this contradicts our assumption that the degree of z^u is not that of a minimal generator of M_2. We thus conclude that $\pi_2(u) = \pi_2(v)$, proving the third part of the lemma.

□

We now introduce some of the conditions on M_1.

Definition 7.2.10. A Graver binomial $x^u - x^v$ of an A-graded ideal M, with $x^u \in M$, and with a given orientation on Δ_1, is called a *sandwich binomial* if the following conditions hold:

1. The support of u consists of more than a singleton.

2. For each $i \in \mathrm{supp}(u)$, $x^{u+v-u_i e_i} \notin M$. There is thus an edge of Δ_1 between any $i \in \mathrm{supp}(u)$ and any $j \in \mathrm{supp}(v)$.

3. In the bipartite digraph consisting of just the oriented edges of Δ_1 that join a vertex in $\mathrm{supp}(u)$ to a vertex in $\mathrm{supp}(v)$, every $i \in \mathrm{supp}(u)$ is either a source or a sink.

4. There is at least one $i \in \mathrm{supp}(u)$ which is a source in this subgraph, and at least one $i \in \mathrm{supp}(u)$ which is a sink.

Note that the first condition is automatically satisfied if the triangulation associated to $\mathrm{conv}(M)$ uses all of the vertices.

Next we further specialize, by choosing a particular matrix A_2, and a particular ideal M_2.

Let A_c be the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & 1 \\ 0 & 0 & 3 & 1 \end{pmatrix}$$

Since this configuration of points is minimally dependent, the Graver basis consists of one element, $y_1y_2y_3 - y_4^3$. There are thus only two A_c-graded monomial ideals. Let $M_c = \langle y_1y_2y_3 \rangle$ be the one corresponding to the triangulation which uses all the vertices. In that case Δ is exactly this triangulation, so we orient the 1-skeleton of Δ as shown in Figure 7.1

Remark 7.2.11. We note by analogy to Lemma 7.2.9 that if $z^u - z^v$ is flippable for $M = \mathrm{Stair}(M_1 \times M_c)$, and $\pi_2(u) \neq \pi_2(v)$ then $y^{\pi_2(u)} - y^{\pi_2(v)}$ is a Graver binomial of M_c. This proof is identical to that in Lemma 7.2.9. Since the unique Graver binomial of M_c is flippable, the corresponding result to part 2 of Lemma 7.2.9 is automatically true.

We now are ready to state our criteria for $\mathrm{Stair}(M_1 \times M_c)$ to have no flips.

Theorem 7.2.12. *Let M_1 be an A_1-graded ideal with no generators of degree 3 or 4 and with the property that for all flippable binomials of M_1, $x^u - x^v$, we have $x_i \notin (M_1 : x^v)$ for all $i \in supp(v)$. Let the associated simplicial complex of M_1, Δ_1, have a locally acyclic*

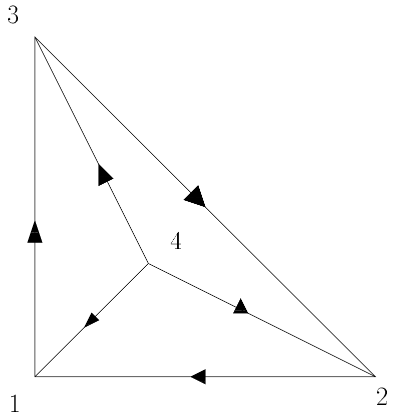

Figure 7.1: The orientation on Δ_c

orientation on its 1-skeleton with no reversible edges, or global sinks, with respect to which all the flippable binomials of M_1 be sandwich binomials. Then $\mathrm{Stair}(M_1 \times M_c)$ *has no flippable binomials.*

Corollary 7.2.13. *If M_1 and A_1 are as above, then the toric Hilbert scheme of $A_1 \times A_c$ is disconnected.*

Proof of Theorem 7.2.12. The proof is in three parts. We first show that no Segre Graver binomial $z^u - z^v$ can be flippable. It then follows from Lemma 7.2.9 and Remark 7.2.11 that if $z^u - z^v$ is a flippable binomial then either $x^{\pi_1(u)} - x^{\pi_1(v)}$ is flippable for M_1 or $y^{\pi_2(u)} - y^{\pi_2(v)}$ is flippable for M_c. We then show that each of these two cases is impossible. These three parts are contained in the three following lemmas. □

Lemma 7.2.14. *Let $z^u - z^v$ be a Segre Graver binomial of $A_1 \times A_c$ with $z^u = z_{ij}z_{kl}$ a minimal generator of $M =$* $\mathrm{Stair}(M_1 \times M_c)$*, where M_1 satisfies the conditions of Theorem 7.2.12, and $z^v = z_{il}z_{kj} \notin M$. Then $z^u - z^v$ is not a flippable binomial of M.*

Proof. Without loss of generality we assume that in the orientation of the one skeleton of Δ_1 and Δ_2 the edge is oriented from k to i, and j to l, and that $i < k$ and $l < j$.

Since the orientation on the one skeleton of Δ_1 has no reversible edges, there must be some simplex of Δ_1 in which i and k are not adjacent. Again after renumbering we may assume that the numbering on this simplex is consistent with the naming of the x variables. So there is some m with $i < m < k$ and $\{i, m, k\} \in \Delta_1$. But now $z_{ij}z_{ml} \in M$, so if $z^u - z^v$

is flippable, $S(z_{ij}z_{ml}, z^u - z^v) = z_{il}z_{ml}z_{kj} \in M$. But since $i < m < k$ and $l < j$, this is not the case. This shows that $z^u - z^v$ is not flippable. □

Lemma 7.2.15. *If $z^u - z^v$ is a flippable binomial of $M = \mathrm{Stair}(M_1 \times M_c)$, where M_1 satisfies the conditions of Theorem 7.2.12, then $\pi_1(u) = \pi_1(v)$*

Proof. We suppose first that $\pi_1(u) \neq \pi_1(v)$, so by Lemma 7.2.9 $x^{\pi_1(u)} - x^{\pi_1(v)}$ is a flippable binomial of M_1. By our assumptions on M_1 this means that $x^{\pi_1(u)} - x^{\pi_1(v)}$ is a sandwich binomial, and that $\pi_2(u) = \pi_2(v)$.

There are now three cases:

1. $|\mathrm{supp}(\pi_2(u))| < 3$ and the vertex V_4 is in $\mathrm{supp}(\pi_2(u))$. Choose $1 \leq j \leq 4$ such that $j \notin \mathrm{supp}(\pi_2(u))$ but $\tau = \mathrm{supp}(\pi_2(u)) \cup \{j\} \in \Delta_c$, with the edge oriented from i to j for all $i \neq j$ in τ. This is always possible.

   By the definition of sandwich binomial, there is some $i \in \mathrm{supp}(\pi_1(u))$ such that i is a source in the induced bipartite subgraph of Δ_1 on $\mathrm{supp}(\pi_1(u))$ and $\mathrm{supp}(\pi_1(v))$. Then, also by the definition, $x^{\pi_1(u)+\pi_1(v)-(\pi_1(u))_i e_i} \notin M_1$, so $\sigma = \mathrm{supp}(\pi_1(u) + \pi_1(v)) \setminus \{i\} \in \Delta_1$. We assume that the ordering on σ induced by the orientation on Δ_1 agrees with the naming of the x variables.

   Let $k = \min_m\{m \in \mathrm{supp}(\pi_1(v))\}$, and choose $l \in \mathrm{supp}(\pi_1(u))$ such that l is a sink in the induced bipartite subgraph of Δ_1 on $\mathrm{supp}(\pi_1(u))$ and $\mathrm{supp}(\pi_1(v))$, so $l < k$. Choose $i \in \pi_2(u)$ such that $z_{li} | z^u$. Now $z_{li}z_{kj} - z_{lj}z_{ki}$, is a Segre Graver binomial with $z_{li}z_{kj} \in M$. Thus since $z^u - z^v$ is flippable, $z^v z_{kj} \in M$. But since we assumed that $x_k \notin (M_1 : x^{\pi_1(v)})$, we have $x^{\pi_1(v)+e_k} \notin M_1$, and also $y^{\pi_2(v)+e_j} \notin M_c$, so there must be some Segre Graver binomial $z^\alpha - z^\beta$ with $z^\alpha \in M$, and $z^\alpha | z^v z_{kj}$. But k and j have been chosen so this is impossible. We thus conclude that if $|\mathrm{supp}(\pi_2(u))| < 3$ with V_4 in $\mathrm{supp}(\pi_2(u))$ then $z^u - z^v$ is not flippable.

2. $|\mathrm{supp}(\pi_2(u))| < 3$ and $V_4 \notin \mathrm{supp}(\pi_2(u))$. As in the previous case we conclude that this is impossible. The proof is very similar, after choosing $j = V_4$, and switching the roles of sources and sinks, and maxima and minima.

3. $|\mathrm{supp}(\pi_2(u))| = 3$. Without loss of generality we assume that $\mathrm{supp}(\pi_2(u)) = \{2, 3, 4\}$. We now claim:

- *If $i \in \operatorname{supp}(\pi_1(u))$ is a source in the induced bipartite subgraph of Δ_1 on the support of $\pi_1(u+v)$ then $u_{ij} > 0$ implies $j = 4$.*

  From the definition of sandwich binomial we know that there is some $k \in \operatorname{supp}(\pi_1(u))$ such that k is a sink in the induced subgraph on $\operatorname{supp}(\pi_1(u+v))$, and that $x^{\pi_1(u)+\pi_1(v)-\pi_1(u)_k e_k} \notin M_1$. Choose j such that $u_{ij} > 0$. We thus have $x^{\pi_1(v)+u_{ij}e_i} \notin M_1$, and $\tau = \operatorname{supp}(\pi_1(v)) \cup \{i\} \in \Delta_1$. We may assume that the ordering on τ induced by the orientation on Δ_1 is the same as the ordering on the x variables. Since M_c is a squarefree ideal it follows that $y^{\pi_2(v)+u_{ij}e_j} \notin M_c$.

  Suppose $z^{v+u_{ij}e_{ij}} \in M$. Since both projections are standard, there must be some Segre Graver binomial $z^\alpha - z^\beta$ with $z^\alpha \in M$ and $z^\alpha | z^{v+u_{ij}e_{ij}}$. Since z^v is standard, and $i > l$ for all $l \in \tau$ with $l \neq i$, we must have $z^\alpha = z_{ij}z_{lm}$, where $l < i$, $j < m$ and $z_{lm}|z^v$. Since $z^u - z^v$ is flippable, $S(z^\alpha, z^u - z^v) = z^{v+e_{lm}} \in M$. Again, by our assumption on M_1, $x^{\pi_1(v)+e_l} \notin M_1$, and since M_c is squarefree $y^{\pi_2(v)+e_m} \notin M_c$. So there must be some Segre Graver binomial $z^\gamma - z^\delta$ with $z^\gamma \in M$ and $z^\gamma | z^{v+e_{lm}}$. But this means $z^\gamma | z^v$, since $z_{lm}|z^v$, implying that $z^v \in M$, a contradiction. So $z^{v+u_{ij}e_{ij}} \notin M$.

  Since $\operatorname{supp}(\pi_2(v)) = \{2, 3, 4\}$, there is some $k \in \operatorname{supp}(\pi_1(v))$ and $l = 4$ such that $v_{kl} > 0$. We know that $k < i$, and since $z^{v+u_{ij}e_{ij}} \notin M$, we must have $l \leq j$. Since 4 is the largest vertex in the set $\{2, 3, 4\}$, it follows that $j = O$.

- *If $i_1 \in \operatorname{supp}(\pi_1(u))$ is a sink in the induced bipartite subgraph of Δ_1 on $i \in \operatorname{supp}(\pi_1(u))$ and $j \in \operatorname{supp}(\pi_1(v))$ then $u_{i_1 j} > 0$ implies $j = 2$.* The proof of this is the same as the above proof with sink and source reversed.

But we have now shown that $\operatorname{supp}(\pi_2(u)) = \{2, 4\}$, contradicting our hypothesis that $|\operatorname{supp}(\pi_2(u))| = 3$. So this case is also impossible.

□

Lemma 7.2.16. *If $z^u - z^v$ is a flippable binomial of $M = \operatorname{Stair}(M_1 \times M_c)$, where M_1 satisfies the conditions of Theorem 7.2.12, then $\pi_2(u) = \pi_2(v)$*

Proof. Suppose $\pi_2(u) \neq \pi_2(v)$. Then it follows from Remark 7.2.11 that $\pi_2(z^u) = y_1y_2y_3$, $\pi_2(z^v) = y_4^3$ and $\pi_1(u) = \pi_1(v)$. Suppose $z_{ik}z_{jl}|z^u$, where the edge in Δ_c between k and l is oriented from l to k, and $i \neq j$. Since z^u is a minimal generator of M, and has degree 3,

$z_{ik}z_{jl} \notin M$, and so we must have the edge in Δ_1 oriented from j to i. As the edge between 4 and l in Δ_c is oriented from 4 to l, it thus follows that $z_{i4}z_{jl} \in M$. But this means that $z_{i4}z^v \in M$, since $z^u - z^v$ is flippable. Since $i \in \mathrm{supp}(v)$ by our assumption on M_1 in the statement of Theorem 7.2.12 we know $x_i x^{\pi_1(v)} \notin M_1$. We also have $y_4 y^{\pi_2(v)} = y_4^4 \notin M_c$. But this means that there must be a Segre Graver binomial $z^\alpha - z^\beta$ with $z^\alpha \in M$ and $z^\alpha | z_{i4}z^v$. As $|\mathrm{supp}(\pi_2(\alpha))| = 2$ for all such Segre Graver binomials, by Lemma 7.2.7, and $\mathrm{supp}(\pi_2(z_{i4}z^v)) = \{4\}$, this is impossible, so we conclude that $i = j$. This means that there is some i such that $\mathrm{supp}(\pi_1(u)) = \{i\}$.

Since there are no global sinks in the orientation on Δ_1, there must be a j such that the edge in Δ_1 is oriented from i to j. This means that $z_{j4}z_{i3} \in M$, so $S(z^u - z^v, z_{j4}z_{i3}) = z^v z_{j4} \in M$ since $z^u - z^v$ is flippable. Since there is an edge in Δ_1 between i and j, we know $x_i x_j \notin M_1$. Since $z^v \notin M$, we also know $x_i^3 \notin M_1$. So if $x_j x_i^3 \in M_1$, one of $x_j x_i^2$ and $x_j x_i^3$ must be a minimal generator of M_1. However our assumption is that M_1 has no generators of degree 3 or 4, so $x_j x_i^3 \notin M_1$, and $y_4^4 \notin M_c$, so as in the above argument we conclude that $z^v z_{j4} \notin M$. From this contradiction we conclude that $\pi_2(u) = \pi_2(v)$. □

Remark 7.2.17. It seems possible that the conditions that M_1 has no generators of degree 3 or 4, and that $x_i \notin (M_1 : x^v)$ for all $i \in \mathrm{supp}(v)$ whenever $x^u - x^v$ is a flippable binomial of M_1 are not necessary. The other conditions seem to be necessary for this approach to work, and have parallels in the Santos proof.

There is some hope that an example might exist in a much smaller number of variables than the Santos example, as there exist simplicial complexes with locally acyclic orientations on their 1-skeleta with no reversible edges and no global sinks on seven vertices.

Example 7.2.18. The following simplicial complex with 21 triangles in 7 variables has a locally acyclic orientation on its 1-skeleton with no reversible edges and no global sinks. The orientation is given by the ordering of the vertices.

$\{6,3,1\}, \{7,1,2\}, \{6,1,5\}, \{1,4,5\}, \{3,1,4\}, \{3,7,1\}, \{1,2,4\}, \{1,5,2\}, \{5,7,2\},$
$\{7,2,6\}, \{2,4,6\}, \{2,3,4\}, \{5,2,3\}, \{2,6,3\}, \{7,6,1\}, \{6,5,3\}, \{5,3,7\}, \{3,4,7\},$
$\{4,7,6\}, \{4,6,5\}, \{4,5,7\}$

This simplicial complex is not a triangulation of any two dimensional polytope, as, for example, the simplices $\{6,3,1\}, \{3,1,4\}$, and $\{3,7,1\}$ all share the edge $\{3,1\}$.

Hundreds of random trials have failed to come up with an example of an A and an A-graded monomial ideal satisfying all the criteria of Theorem 7.2.12. The main obstacle appears to be that if all the generators of the monomial ideal are not squarefree, Δ will be a simplex, so only have one locally acyclic orientation. It appears to be difficult to construct ideals with sufficiently interesting Δ. Also, the requirement that $x_i \notin (M_1 : x^v)$ for all $i \in \text{supp}(v)$ whenever $x^u - x^v$ is a Graver binomial with x^u a minimal generator and x^v standard becomes harder to satisfy if all generators are of low degree. Perhaps the greatest hope for this approach to be successful is to first obtain a characterization of when an ideal is a monomial A-graded ideal for some A, and then use this to build the desired example.

It should be noted that if Δ is a triangulation satisfying the conditions of Santos' Corollary 2.7, and its Stanley-Reisner ideal $I(\Delta)$ is A-graded, then $I(\Delta)$ satisfies the conditions of Theorem 7.2.12.

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