INTRODUCTION TO TROPICAL ALGEBRAIC GEOMETRY

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These notes are the lecture notes from my lectures on tropical geometry at the ELGA 2011 school on Algebraic Geometry and Applications in Buenos Aires August 1-5 2011. Please send any typos to me at `D.Maclagan@warwick.ac.uk`. More details can be found in the draft book *Introduction to tropical algebraic geometry*, with Bernd Sturmfels, which is available at `www.warwick.ac.uk/staff/D.Maclagan/papers/TropicalBook.pdf`. Comments and corrections about the draft are *very* welcome.

1. Lecture 1

Tropical algebraic geometry is algebraic geometry over the tropical semiring. It replaces a variety by its combinatorial shadow.

Outline of the lectures:

(1) Introduction
(2) Fundamental theorem and structure theorem
(3) Computing tropical varieties
(4) Connections to toric varieties
(5) Enumerative geometry and other applications.

1.1. Tropical arithmetic.

Definition 1.1. The tropical semiring is $\mathbb{R} \cup \{\infty\}$, with operation $\oplus$ and $\otimes$ given by $a \oplus b = \min(a, b)$ and $a \otimes b = a + b$. It is associative and distributative, with additive identity ∞ and multiplicative identity 0. This satisfies every axiom of a ring except additive inverses, so is a semiring.

Tropical mathematics has existed for much longer than tropical geometry, and has seen use in semigroup theory and optimization. The name "tropical" was coined by some French mathematicians in honour of the Brazilian computer scientists Imre Simon.

Tropical operations are often simpler than regular operations. For example we have the "Freshman's dream": $(x + y)^n = x^n + y^n$.

Tropical polynomials are piecewise linear functions.

Example 1.2.
(1) The tropical polynomial $F(x) = -2 \otimes x^3 \oplus -1 \otimes x^2 \oplus 1 \otimes x \oplus 5$ is $\min(3x - 2, 2x - 1, x + 1, 5)$ in regular arithmetic. This is the piecewise linear function whose graph is shown on the left in Figure 1.
(2) The tropical multivariate polynomial $x \oplus y \oplus 0$ is the piecewise linear function $\min(x, y, 0)$ in regular arithmetic. Note that the zero cannot be removed here, as zero is not the additive identity. This is a function from $\mathbb{R}^2$ to $\mathbb{R}$ whose domain is shown in Figure 2

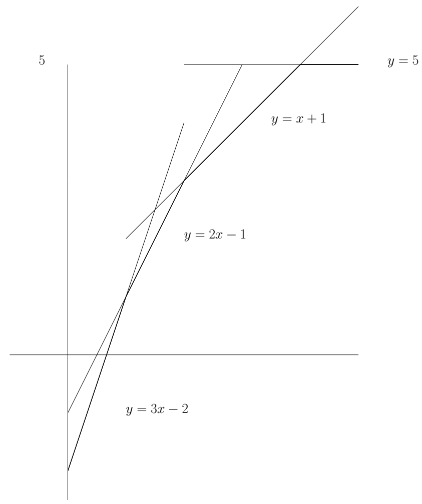

FIGURE 1. A tropical polynomial

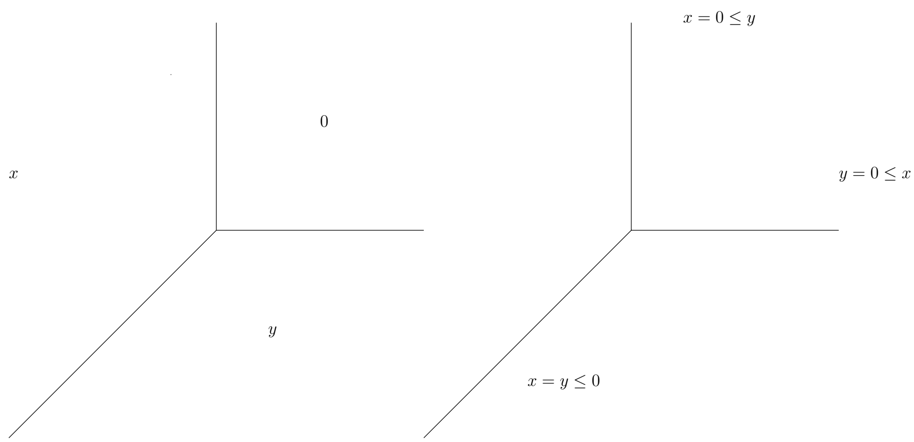

FIGURE 2. A tropical line

With no subtraction, it is not obvious how to solve polynomial equations. For example, the equation $x \oplus 2 = 5$ has no solution. This problem has the following resolution.

Definition 1.3. The hypersurface $V(F))$ defined by the tropical polynomial F in n variables is the nonlinear locus of F. This is the locus in $\mathbb{R}^n$ where F is not differentiable, or equivalently the $x \in \mathbb{R}^n$ for which the minimum is achieved twice in $F(x)$

Example 1.4. In the first example of Example 1.2 $V(F) = \{0, 3, 4\}$. In the second it is the union of the three rays shown on the right in Figure 2.

Example 1.5. The tropical quadratic formula is particularly simple. If $F(x) = a \otimes x^2 \oplus b \otimes x \oplus c$, then the graph of F is shown in Figure 3. Note that there are two cases, depending on the sign of the tropical discriminant $a+c-2b$. If $2b \leq a+c$ then $V(F) = \{c-b, b-a\}$. If $2b \geq a+c$ then $V(F) = \{(c-a)/2\}$. Compare this with the

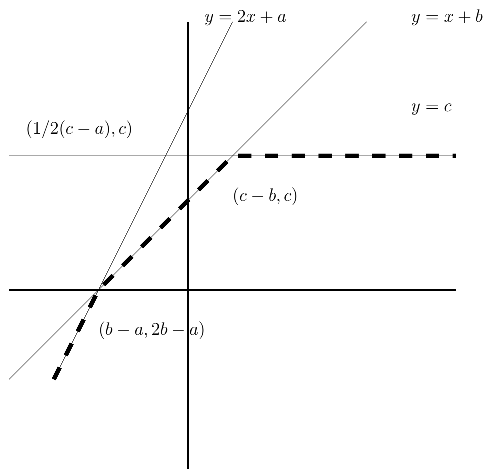

FIGURE 3. Tropical quadratic polynomials

usual quadratic formula, and the usual discriminant. Note also that it much easier to solve higher degree polynomials!

Goal of first half of the week: Develop the theory of tropical varieties and understand their structure and connection with "classical" varieties.

There are several approaches to tropical geometry. We will follow the "embedded approach", which focuses on tropicalizing classical varieties. There is also an approach which focuses on developing an abstract theory of tropical varieties in their own right (see work of Mikhalkin and collaborators [Mik], [Mik06]). This is most developed for curves, and connects best to classical varieties with additional niceness conditions.

Tropical geometry has become a broad field, with connections to optimization, integrable systems, mirror symmetry,

1.2. Connection with classical algebraic geometry.

For a field K we set $K^* = K \setminus \{0\}$. Fix a valuation $\text{val} : K^* \to \mathbb{R}$. This is a function satisfying:

(1) $\text{val}(ab) = \text{val}(a) + \text{val}(b)$
(2) $\text{val}(a + b) \geq \min(\text{val}(a), \text{val}(b))$.

Example 1.6. (1) $K = \mathbb{C}$ with the trivial valuation $\text{val}(a) = 0$ for all $a \in \mathbb{C}^*$

(2) $K = \mathbb{C}\{\{t\}\}$, the field of Puiseux series. This is $\cup_{n \geq 1} \mathbb{C}((t^{1/n}))$. It is the algebraic closure of the field of Laurent series. Elements are Laurent series with rational exponents where in any given series the exponents all have a common denominator. A valuation is given by taking $a \in \mathbb{C}\{\{t\}\}$ to the lowest exponent appearing. For example, $\text{val}(3t^{-1/2} + 8t^2 + 7t^{13/3} + \dots) = -1/2$.

(3) $K = \mathbb{Q}$ or $\mathbb{Q}_p$ with the p-adic valuation. If $q = p^n a/b \in \mathbb{Q}$ with $p \not| a, b$ then $\text{val}_p(q) = n$. For example, $\text{val}_2(8) = 3$, and $\text{val}_3(5/6) = -1$.

Definition 1.7. The tropicalization of a Laurent polynomial $f = \sum c_u x^u \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}$ is $\text{trop}(f) : \mathbb{R}^n \to \mathbb{R}$ given by

$$\text{trop}(f)(w) = \min(\text{val}(c_u) + w \cdot u).$$

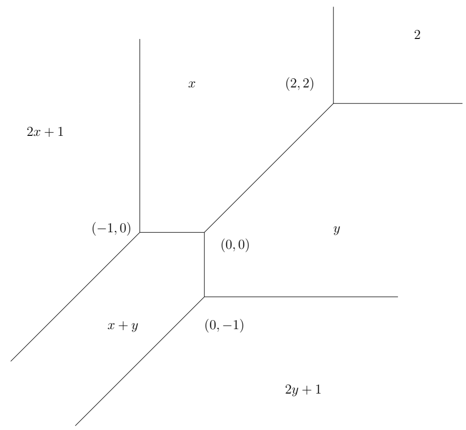

FIGURE 4. A tropical quadric

This is obtained by regarding the addition and multiplication as tropical addition and multiplication, and changing the coefficients to their valuations.

Example 1.8. Let $K = \mathbb{Q}$ with the 2-adic valuation, and let $f = 6x^2 + 5xy + 10y^2 + 3x - y + 4 \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$. Then $trop(f) = \min(2x+1, x+y, 2y+1, x, y, 2)$. This is illustrated in Figure 4.

For $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}$ the (classical) hypersurface $V(f)$ equals $\{x \in (K^*)^n : f(x) = 0\}$. The *tropicalization* $\operatorname{trop}(V(f))$ of $V(f)$ is the tropical hypersurface of $\operatorname{trop}(f)$. This is the nondifferentiability locus of $\operatorname{trop}(f)$, or equivalently

$$\operatorname{trop}(V(f)) = \{w \in \mathbb{R}^n : \text{ the minimum in } \operatorname{trop}(f)(w) \text{ is achieved at least twice } \}.$$

Note that $\operatorname{trop}(x^u f) = \operatorname{trop}(x^u) + \operatorname{trop}(f)$, so $\operatorname{trop}(x^u f)(w) = w \cdot u + \operatorname{trop}(f)(w)$, and thus $\operatorname{trop}(V(x^u f)) = \operatorname{trop}(f)$. This explains why the naturatl place to consider tropical varieties is $(K^*)^n$, not $\mathbb{A}^n$ or $\mathbb{P}^n$.

Let $Y = V(I)$ be a subvariety of $T = (K^*)^n$. If $I = \langle f_1, \dots, f_r \rangle$ then

$$\begin{aligned} Y = V(f_1, \dots, f_r) &= \{x \in T : f_1(x) = \dots = f_r(x) = 0\} \\ &= \{x \in T : f(x) = 0 \text{ for all } f \in I\} \end{aligned}$$

Definition 1.9. The tropicalication of a variety $Y \subseteq T$ is

$$\operatorname{trop}(Y) = \bigcap_{f \in I(Y)} \operatorname{trop}(V(f)).$$

Example 1.10.

(1) $Y = V(x+y+z+w, x+2y+5z+11w) \subseteq (\mathbb{C}^*)^4$. Then $\operatorname{trop}(Y)$ has the property that if $x \in \operatorname{trop}(Y)$, then $x + \lambda(1,1,1,1) \in \operatorname{trop}(Y)$ for all $\lambda \in \mathbb{R}$. We can thus quotient by the span of $(1,1,1,1)$ and describe $\operatorname{trop}(Y)$ in $\mathbb{R}^4/\mathbb{R}(1,1,1,1) \cong \mathbb{R}^3$. This consists of four rays, being the images of the positive coordinate directions $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, and $(0,0,0,1)$.

(2) Let $Y = V(t^3x^3 + x^2y + xy^2 + t^3y^3 + x^2 + t^{-1}xy + y^2 + x + y + t^3) \subseteq (\mathbb{C}\{\{t\}\}^*)^2$. Then $trop(V(f))$ is shown in Figure 5. This is a "tropical elliptic curve".

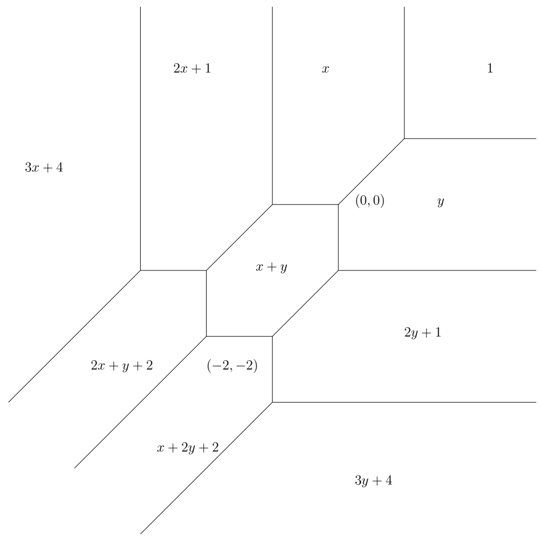

FIGURE 5.

Warning: If $Y = V(f_1, \dots, f_r)$, $\text{trop}(Y)$ does not always equal $\cap_{i=1}^r \text{trop}(V(f_i))$. For example, in the first part of Example 1.10, the tropicalication of both generators is the same set, which is larger than the tropical variety.

Definition 1.11. If $Y = V(I)$, a set $f_1, \dots, f_r \in I$ with

$$\text{trop}(Y) = \bigcap_{i=1}^{r} \text{trop}(V(f_i))$$

is called a tropical basis for I. Finite tropical bases always exist, so $\text{trop}(Y)$ is a piecewise linear object.

Guiding question: Which properties of Y or of compactifications of Y can be deduced from $\text{trop}(Y)$?

2. Exercises

(1) Show that if $\text{val} : K^* \to \mathbb{R}$ is a valuation, and $\text{val}(a) \neq \text{val}(b)$, then $\text{val}(a+b) = \min(\text{val}(a), \text{val}(b))$.
(2) Give formulas to solve the tropical cubic.
(3) Draw the tropical varieties $\text{trop}(V(f))$ for the following $f \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$.
   (a) $f = t^3x + (t + 3t^2 + 5t^4)y + t^{-2}$;
   (b) $f = (t^{-1} + 1)x + (t^2 - 3t^3)y + 5t^4$;
   (c) $f = t^3x^2 + xy + ty^2 + tx + y + 1$;
   (d) $f = 4t^4x^2 + (3t + t^3)xy + (5 + t)y^2 + 7x + (-1 + t^3)y + 4t$;
   (e) $f = tx^2 + 4xy - 7y^2 + 8$;
   (f) $f = t^6x^3 + x^2y + xy^2 + t^6y^3 + t^3x^2 + t^{-1}xy + t^3y^2 + tx + ty + 1$.
(4) Let $f = ax + by + c$, where $a, b, c \in \mathbb{C}\{\{t\}\}$. What are the possibilities for $\text{trop}(V(f))$? How does this change if $\mathbb{C}\{\{t\}\}$ is changed to $\mathbb{Q}$ with the p-adic valuation? Such tropical varieties are called tropical lines. (Harder) Repeat this question for $f = ax^2 + bxy + cy^2 + dx + ey + f$ (tropical quadrics).

(5) Show that any two general tropical lines in $\mathbb{R}^2$ intersect in a unique point, and there is a unique tropical line containing any two general points in $\mathbb{R}^2$. What are the notions of genericity here?

(6) (Much harder) Show that if f and g are two general polynomials in $K[x^{\pm 1}, y^{\pm 1}]$ of degrees d and e respectively then $\operatorname{trop}(V(f)) \cap \operatorname{trop}(V(g))$ consists of at most de points. This can be refined by adding multiplicities to make the intersection consist of exactly de points counted with multiplicity (and more generally to n general polynomials in n variables). This is the tropical Bézout theorem.

3. Lecture 2

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In the previous lecture we defined the tropical variety corresponding to an ideal $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ to be the set of all points in $\mathbb{R}^n$ that are in the tropical hypersurfaces $\operatorname{trop}(V(f))$ for all $f \in I$, so are the common "tropical zeros" of the tropicalizations of the polynomials f. The connection between the tropicalization $\operatorname{trop}(Y)$ of a variety $Y = V(I) \subseteq (K^*)^n$ and the original variety Y is closer than this analogy might suggest, as the following theorem shows.

Theorem 3.1 (Fundamental theorem of tropical algebraic geometry)**.** *Let K be an algebraically closed field with a nontrivial valuation $\operatorname{val} : K^* \to \mathbb{R}$, and let Y be a subvariety of $(K^*)^n$. Then*

$$\begin{aligned}\operatorname{trop}(Y) &= \operatorname{cl}(\operatorname{val}(Y)) \\ &= \operatorname{cl}((\operatorname{val}(y_1), \dots, \operatorname{val}(y_n)) : y = (y_1, \dots, y_n) \in Y),\end{aligned}$$

where the closure is in the usual Euclidean topology on $\mathbb{R}^n$.

Example 3.2. Let $Y = V(x + y + 1) \subseteq (K^*)^2$, where $K = \mathbb{C}\{\{t\}\}$. Then $Y = \{(a, -1 - a) : a \in \Bbbk^* \setminus \{-1\}\}$. Note that

$$(\operatorname{val}(a), \operatorname{val}(-1-a)) = \begin{cases} (\operatorname{val}(a), 0) : & \operatorname{val}(a) > 0 \\ (\operatorname{val}(a), \operatorname{val}(a)) : & \operatorname{val}(a) < 0 \\ (0, \operatorname{val}(b)) : & a = -1 + b, \operatorname{val}(b) > 0 \\ (0, 0) : & \text{otherwise} \end{cases}$$

Note that, as predicted by Theorem 3.1, the union of these sets is precisely $\operatorname{trop}(Y)$. This is illustrated in Figure 6

Example 3.3. Let $f = 4x^2 + xy - 4y^2 + x - y + 4 \in \mathbb{Q}[x, y]$, where $\mathbb{Q}$ has the 2-adic valuation. Then $\operatorname{trop}(V(f))$ is shown in Figure 7. Note that the point $(2, 2) \in V(f)$, and $(\operatorname{val}(2), \operatorname{val}(2)) = (1, 1) \in \operatorname{trop}(V(f))$.

The hard part of Theorem 3.1 is to show that if $w \in \operatorname{trop}(Y) \cap (\operatorname{im} \operatorname{val})^n$ then there is $y \in Y$ with $\operatorname{val}(y) = w$. Showing that all $\operatorname{val}(y)$ for $y \in Y$ live in $\operatorname{trop}(Y)$ is comparatively easy.

The slogan form of Theorem 3.1 is then:

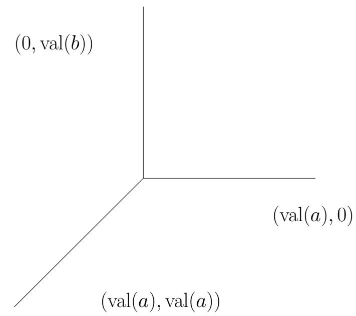


FIGURE 6. The Fundamental Theorem applied to a tropical line

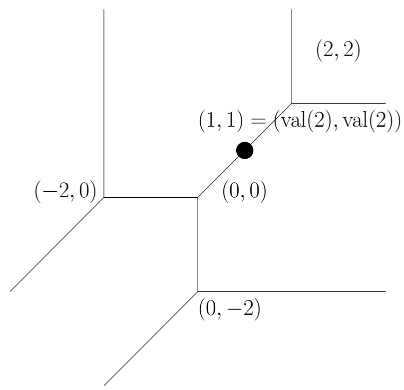


FIGURE 7. An example of the Fundamental Theorem

Tropical varieties are combinatorial shadows of classical varieties.

The word "combinatorial" is justified by the Structure Theorem for tropical varieties, which gives combinatorial contraints on which sets can be tropical varieties. We first recall some definitions.

Definition 3.4. A variety $Y \subseteq (K^*)^n$ is irreducible if we cannot write $Y = Y_1 \cup Y_2$ for $Y_1, Y_2 \subsetneq Y$ subvarieties of Y.

Note that by Theorem 3.1 we have $\operatorname{trop}(Y_1 \cup Y_2) = \operatorname{trop}(Y_1) \cup \operatorname{trop}(Y_2)$.

Definition 3.5. A polyhedron in $\mathbb{R}^n$ is the intersection of finitely many half-spaces in $\mathbb{R}^n$. This can be written as:

$$P = \{x \in \mathbb{R}^n : Ax \leq b\},$$

where A is a $d \times n$ matrix, and $b \in \mathbb{R}^d$. The dimension of P is the dimension of the subspace $\ker(A)$. For a subgroup $\Gamma \subseteq \mathbb{R}$, we say that a polyhedron P is Γ-rational if A has rational entries, and $b \in \Gamma^d$. When $\Gamma = \mathbb{Q}$, we say that polyhedron is rational.

If $b = 0$, then P is called a cone. In that case there are $\mathbf{v}_1, \ldots, \mathbf{v}_s$ for which $P = \operatorname{pos}(\mathbf{v}_1, \ldots, \mathbf{v}_s) := \{\sum_{i=1}^s \lambda_i \mathbf{v}_i : \lambda_i \geq 0\}$.

A face of a polyhedron P determined by $w \in (\mathbb{R}^n)^*$ is the set

$$\operatorname{face}_w(P) = \{x \in P : w \cdot x \leq w \cdot y \text{ for all } y \in P\}.$$

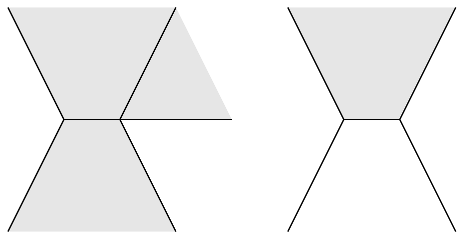


FIGURE 8. The complex on the left is pure, while the one on the r ight is not.

Example 3.6. Let P be the square with vertices $\{(0,0),(1,0),(0,1),(1,1)\}$. This has the description

$$P = \left\{ x \in \mathbb{R}^2 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x \leq \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Then we have

(1) $\text{face}_{(1,0)}(P)$ is the edge of the square with vertices $\{(0,0),(0,1)\}$,
(2) $\text{face}_{(1,1)}(P)$ is the vertex $(0,0)$, and
(3) $\text{face}_{(0,0)}(P)$ is P.

Definition 3.7. A polyhedral complex Σ is a finite union of polyhedra for which any nonempty intersection of two polyhedra $\sigma_1, \sigma_2 \in \Sigma$ is a face of each. If every polyhedron in Σ is a cone, then Σ is called a fan. The support of Σ is the set

$$|\Sigma| = \{x \in \mathbb{R}^n : x \in \sigma \text{ for some } \sigma \in \Sigma\}.$$

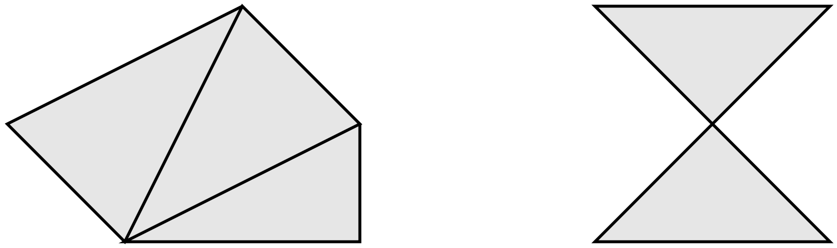
Definition 3.8. A polyhedral complex is *pure* if the dimension of every maximal polyhedron is the same. See Figure 8.

Definition 3.9. The lineality space L of a polyhedral complex Σ is the largest subspace of $\mathbb{R}^n$ for which $x + l \in \Sigma$ for all $x \in \Sigma, l \in L$.

Definition 3.10. A weighted polyhedral complex is a polyhedral complex Σ with a weight $w_\sigma \in \mathbb{N}$ for all maximal-dimensional $\sigma \in \Sigma$.

Let Σ be a weighted im val-rational polyhedral complex that is pure of dimension d. The complex Σ is *balanced* if the following "zero-tension" conditions hold.

(1) If Σ is a one-dimensional rational fan, let $\mathbf{u}_1, \dots, \mathbf{u}_s$ be the first lattice points on the rays of Σ, and let w_i be the weight of the cone containing the lattice point $\mathbf{u}_i$. Then Σ is balanced if $\sum_{i=1}^s w_i \mathbf{u}_i = 0$.
(2) For other Σ, fix a $(d-1)$-dimensional polyhedron τ of Σ. Let $L = \text{span}(x - y : x, y \in \tau)$ be the affine span of τ. Let $\text{star}_\Sigma(\tau)$ be the rational polyhedral fan whose support is $\{w \in \mathbb{R}^n : \text{ there exists } \epsilon > 0 \text{ for which } w' + \epsilon w \in \Sigma \text{ for all } w' \in \tau\} + L$. This has one cone for each polyhedron $\sigma \in \Sigma$ that contains τ, and has lineality space L. The quotient $\text{star}_\Sigma(\tau)/L$ is a one dimensional fan. We say that Σ is balanced at τ if the one-dimensional fan $\text{star}_\Sigma(\tau)/L$ is balanced. The polyhedral complex Σ is balanced if Σ is balanced at all $(d-1)$-dimensional cones.



Connected in codimension one   Not connected in codimension one

FIGURE 9.

Example 3.11. Let $f = x^2y^2 + x^3 + y^3 + 1 \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$. Then $\operatorname{trop}(V(f))$ is a one-dimensional fan with four rays: $\operatorname{pos}((1,0))$, $\operatorname{pos}((0,1))$, $\operatorname{pos}((-2,-1))$, and $\operatorname{pos}((-1,-2))$, with weights $3, 3, 1$ and 1. This is balanced as $3(1,0) + 3(0,1) + 1(-2,-1) + 1(-1-2) = (0,0)$.

Definition 3.12. A pure polyhedral complex is connected through codimension-one if the graph with a vertex for each maximal polyhedra $\sigma \in \Sigma$ and an edge between two vertices if the corresponding polyhedra intersect in a codimension-one face. For example, the polyhedral complex on the left of Figure 9 is connected through codimension-one, while the one on the right is not.

Theorem 3.13. *Let Y be a d-dimensional irreducible subvariety of $(K^*)^n$. Then $\operatorname{trop}(Y)$ is the support of a pure d-dimensional weighted balanced $(\operatorname{im}\operatorname{val})$-rational polyhedral complex that is connected through codimension-one.*

This means that tropical varieties have a discrete structure, and record information about the original variety (such as its dimension). We will later see some other features that retained by a tropical variety.

4. Exercises

(1) Show that if K is an algebraically closed field with a nontrivial valuations $\operatorname{val} : K^* \to \mathbb{R}$ (ie there is $a \in K^*$ with $\operatorname{val}(a) \neq 0$) then $\operatorname{im}\operatorname{val}$ is dense in $\mathbb{R}$.

(2) Verify the fundamental theorem of tropical algebraic geometry and the structure theorem for $Y = V(f)$ for the following polynomials $f \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$:
   (a) $f = 3x + t^2y + 2t$;
   (b) $f = tx^2 + xy + ty^2 + x + y + t$;
   (c) $f = x^3 + y^3 + 1$.

(3) (Harder, but not impossible) Prove the easy direction of the fundamental theorem.

(4) Let $Y = \operatorname{trop}(V(x+y+z+1)) \subseteq (\mathbb{C}^*)^3$. Compute $\operatorname{trop}(Y)$, and a polyhedral complex Σ with support $\operatorname{trop}(Y)$. Show that Σ is balanced if we put the weight one on each top-dimensional cone.

(5) Let Σ be the pure one-dimensional polyhedral fan with cones $\operatorname{pos}((1,0))$, $\operatorname{pos}((0,1))$, $\operatorname{pos}((-1,1))$, $\operatorname{pos}((-1,-1))$. Find all weights $w \in \mathbb{N}^4$ for which Σ is balanced.

5. Lecture 3

An important aspect of tropical varieties, which we now discuss, is that they can actually be computed in practice. This uses an extension of the theory of Gröbner bases to fields with a valuation.

Fix a splitting $\operatorname{im}\operatorname{val} \to K^*$ of the valuation. This is a group homomorphism $u \mapsto t^u$ with $\operatorname{val}(t^u) = u$. For example, when $K = \mathbb{C}$ with the trivial valuation ($\operatorname{val}(a) = 0$ for all $a \neq 0$), then we can choose the splitting $0 \mapsto 1$. When $K = \mathbb{C}\{\{t\}\}$, we can choose $u \mapsto t^u$, and when $K = \mathbb{Q}$ with the p-adic valuation, we can choose $u \mapsto p^u$.

Let $R = \{a \in K : \operatorname{val}(a) \geq 0\}$ be the valuation ring of K. The ring R is local, with maximal ideal $\mathfrak{m} = \{a \in K : \operatorname{val}(a) > 0\} \cup \{0\}$. The quotient $\Bbbk = R/\mathfrak{m}$ is the residue field.

Example 5.1. (1) When $K = \mathbb{C}$ has the trivial valuation, we have $R = \mathbb{C}$, and $\mathfrak{m} = 0$, so $\Bbbk = \mathbb{C}$.
(2) When $K = \mathbb{C}\{\{t\}\}$, $R = \bigcup \mathbb{C}[[t^{1/n}]]$, and $\Bbbk = \mathbb{C}$.
(3) When $K = \mathbb{Q}_p$, $R = \mathbb{Z}_p$, and $\Bbbk = \mathbb{Z}/p\mathbb{Z}$.

Given a polynomial $f = \sum c_u x^u \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, and $w \in (\operatorname{im}\operatorname{val})^n$, the initial form is

$$\operatorname{in}_w(f) = \sum_{\operatorname{val}(c_u) + w\cdot u = \operatorname{trop}(f)(w)} \overline{t^{-\operatorname{val}(c_u)} c_u} x^u \in \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}],$$

where for $a \in R$ we denote by $\overline{a}$ the image of a in $\Bbbk$.

Example 5.2. Let $K = \mathbb{Q}$ with the 2-adic valuation, and let $f = 2x^2 + xy + 6y^2 + 5x - 3y + 4$. Then for $w = (2,2)$ we have $\operatorname{trop}(f)(w) = 2$, so $\operatorname{in}_w(f) = \overline{5}x + \overline{-3}y + \overline{2^{-2}} = x + y + 1 \in \mathbb{Z}/2\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$.

For $w = (-2,-1)$ we have $\operatorname{trop}(f)(w) = -3$, so $\operatorname{in}_w(f) = x^2 + xy$.

Given an ideal $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $w \in (\operatorname{im}\operatorname{val})^n$q, the initial ideal of I is

$$\operatorname{in}_w(I) = \langle \operatorname{in}_w(f) : f \in I \rangle.$$

As for usual Gröbner bases, the initial ideal need not be generated by the initial forms of a generating set for I, but there are always finite generating sets for which this is the case. These finite sets (Gröbner bases) can be computed using a variant of the standard Gröbner basis algorithm.

Proposition 5.3. *Let $Y = V(I) \subseteq (K^*)^n$ and let $w \in (\operatorname{im}\operatorname{val})^n$. Then $w \in \operatorname{trop}(Y)$ if and only if $\operatorname{in}_w(I) \neq \langle 1 \rangle \subseteq \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.*

Example 5.4. Let $I = \langle 2x^2 + xy + 6y^2 + 5x - 3y + 4 \rangle \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$, where $\mathbb{Q}$ has the 2-adic valuation. The claim of Proposition 5.3 is illustrated in Figure 10.

There is a polyhedral complex Σ with $\operatorname{in}_w(I)$ constant for $w \in \operatorname{relint}(\sigma)$ for any $\sigma \in \Sigma$. This is called the Gröbner complex of I. Proposition 5.3 implies that $\operatorname{trop}(Y)$ is the union of the polyhedra σ in the Gröbner complex of $I(Y)$ for which $\operatorname{in}_w(I) \neq \langle 1 \rangle$ for any $w \in \operatorname{relint}(\sigma)$.

The software `gfan` [Jen] by Anders Jensen computes tropical varieties by exploiting this Gröbner description.

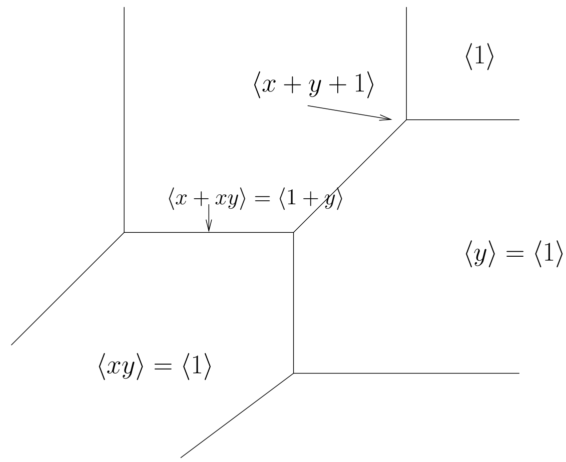


FIGURE 10. The Gröbner complex of Example 5.4

Recall that $\mathrm{trop}(Y)$ is the support of a balanced weighted polyhedral complex Σ. We assume, as described above, that $\mathrm{in}_w(I(Y))$ is constant on the relative interior of cones of Σ.

The weights $w_\sigma \in \mathbb{N}$ on maximal polyhedra in Σ balanced are defined as follows. Fix $w \in (\mathrm{im}\,\mathrm{val})^n$ in the relative interior of a maximal polyhedron of Σ. Then $V(\mathrm{in}_w(I)) \subseteq (\Bbbk^*)^n$ is a union of $(\Bbbk^*)^d$-orbits. We set w_σ to be the number of such orbits (counted with multiplicity).

Hidden in the proof of the Structure Theorem (Theorem 3.13) is the fact that this choice makes the polyhedral complex Σ balanced.

Example 5.5. Let $f = x^2+3x+2+x^2y+2xy^2-2y^2 \subseteq \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$. Then $\mathrm{trop}(V(f))$ is a one-dimensional fan with five rays, spanned by the vectors $\{(1,0),(0,1),(-1,0),(-1,-1),(0,-1)\}$. When $w = (0,1)$, $\mathrm{trop}(f)(w) = 0$, so $\mathrm{in}_w(f) = x^2+3x+2 = (x+2)(x+1)$. thus $V(\mathrm{in}_w(f)) = \{(-2,a) : a \in \mathbb{C}^*\} \cup \{(-1,a) : a \in \mathbb{C}^*\}$, so the weight on the cone spanned by $(0,1)$ is 2. When $w = (-1,-1)$, $\mathrm{in}_w(f) = x^2y + 2xy^2$, so $V(\mathrm{in}_w(f)) = V(x+2y) = \{(2a,-a) : a \in \mathbb{C}^*\}$, and thus the weight on this cone is one. Similarly, the cones on the cone spanned by $(1,0)$ is two, and all other weights are one. Note that

$$2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} -1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} -1 \\ -1 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 10 \end{pmatrix}.$$

This is illustrated in Figure 11.

Drawing curves in the plane

Let $C = V(f) \subseteq (K^*)^2$ for $f \in K[x^{\pm 1}, y^{\pm 1}]$. We've seen that $\mathrm{trop}(C)$ is a weighted balanced one-dimensional polyhedral complex. We now discuss how to draw $\mathrm{trop}(C)$.

Case one: The valuation on K is trivial ($\mathrm{val}(a) = 0$ for all $a \neq 0$). Let $f = \sum c_u x^u$, and let P be the Newton polytope of f. This is the convex hull of the $u \in \mathbb{Z}^2$ with $c_u \neq 0$: $P = \{\sum \lambda_u u : c_u \neq 0, \sum \lambda_u = 1\}$. The *normal fan* to P is the fan $N(P)$ with cones $C[w] = \mathrm{cl}(w' : \mathrm{face}'_w(P) = \mathrm{face}_w(P))$.

Example 5.6. Let $f = x^2y + 5y^2 - 3x + 2$. Then P and $N(P)$ are illustrated in Figure 12.

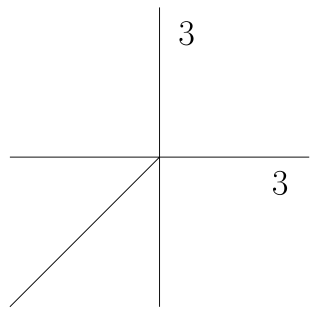


FIGURE 11.

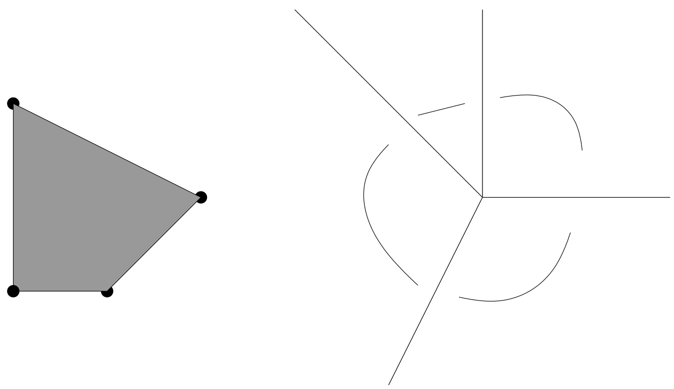


FIGURE 12.

In this case trop(C) is the union of all one-dimensional cones in the normal fan $N(P)$. For a one-dimensional cone $\sigma \in \text{trop}(C)$, the weight w_σ is the lattice length of the edge of P with normal vector in σ. The lattice length is the number of lattice points (elements of $\mathbb{Z}^2$) in the edge minus one.

Case 2 K has a nontrivial valuation. Let $\widetilde{P}$ be the convex hull of the set $\{(u, \text{val}(c_u)) : c_u \neq 0\}$ in $\mathbb{R}^{2+1} = \mathbb{R}^3$, and let $N(\widetilde{P})$ be its normal fan. The regular subdivision $\Delta_{(\text{val}(c_u))}$ of P corresponding to the vector $(\text{val}(c_u))$ is the projection of P to the "lower faces" of $\widetilde{P}$.

Example 5.7. Let $f = 2x^2 + xy - 6y^2 + 5x - 3y + 2 \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$ where $\mathbb{Q}$ has the 2-adic valuation. Then the regular subdivision of the Newton polytope of f corresponding to $\text{val}(c_u)$ is shown in Figure 13

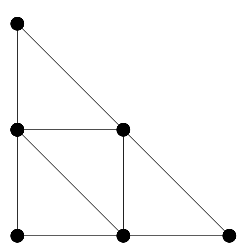


FIGURE 13. The regular subdivision induced by $(\text{val}(c_u))$

In this case $\text{trop}(C) = \{w \in \mathbb{R}^2 : \text{face}_{(w,1)}(\widetilde{P}) \text{ is not a vertex }\}$. This is the dual graph to $\Delta_{(\text{val}(c_u))}$.

Example 5.8. Let $f = 27x^3+6x^2y+12xy^2+81y^2+3x^2+5xy+3y^2+3x+2y+243 \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$ where $\mathbb{Q}$ has the 3-adic valuation.

Then the regular triangulation corresponding to $(\text{val}(c_u))$ is shown in Figure 14, along with the tropical variety.

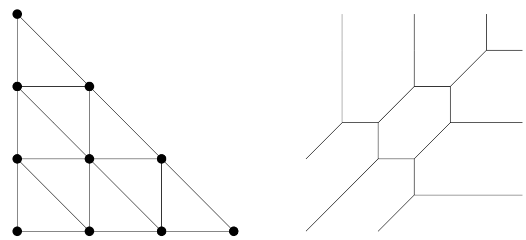


FIGURE 14.

6. Exercises

(1) Show that the residue field of $\mathbb{C}\{\{t\}\}$ is $\mathbb{C}$. Show that the residue field of $\mathbb{Q}$ with the p-adic valuation is $\mathbb{Z}/p\mathbb{Z}$.
(2) Show that if K is an algebraically closed field with a valuation, then the residue field is algebraic closed.
(3) Let $f = 8x^2 + xy + 12y^2 + 3 \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$. Compute all initial ideals $\text{in}_w(I)$ of $I = \langle f \rangle$ as w varies when
  (a) $\mathbb{Q}$ has the trivial valuation,
  (b) $\mathbb{Q}$ has the 2-adic valuation, and
  (c) $\mathbb{Q}$ has the 3-adic valuation.
(4) Show that if K has the trivial valuation, then the Gröbner complex is a fan, so every polyhedron is a cone.
(5) Go back to the plane curves of the first exercise set and draw them using the regular triangulation method.

7. Lecture 4

In this lecture we cover some of the connections between tropical geometry and toric varieties. We assume here that $K = \mathbb{C}$. This means that for $Y \in (\mathbb{C}^*)^n$, $\text{trop}(Y)$ is the support of a weighted balanced rational polyhedral fan.

Definition 7.1. A (normal) toric variety is a normal variety X containing a dense copy of $T = (\mathbb{C}^*)^n$ with an action of T on X that extends the action of T on itself.

Examples include:

(1) $X = (\mathbb{C}^*)^n$
(2) $X = \mathbb{A}^n \subset (\mathbb{C}^*)^n = \{x \in \mathbb{A}^n : x_i \neq 0 \text{ for } 1 \leq i \leq n\}$.
(3) $X = \mathbb{P}^n \subset (\mathbb{C}^*)^n = \{x \in \mathbb{P}^n : x_i \neq 0 \text{ for } 0 \leq i \leq n\}$.
(4) $X = \mathbb{P}^1 \times \mathbb{P}^1$

A toric variety X is a union of T-orbits. These can be recorded using a polyhedral fan Σ.

Example 7.2. The projective plane $\mathbb{P}^2$ decomposes into the following $T = (\mathbb{C}^*)^2$-orbits:

$$(\mathbb{C}^*)^2 \cup \{[0 : a : b] : a, b \in \mathbb{C}^*\} \cup \{[a : 0 : b] : a, b \in \mathbb{C}^*\} \cup \{[a : b : 0] : a, b \in \mathbb{C}^*\}$$
$$\cup \{[1 : 0 : 0]\} \cup \{[0 : 1 : 0]\} \cup \{0 : 0 : 1]\}.$$

The corresponding fan is shown in Figure 15.

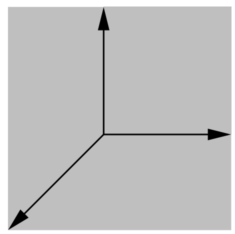


FIGURE 15. The fan of $\mathbb{P}^2$

Alternatively (and more standardly), given a rational polyhedral fan Σ we construct a toric variety X_Σ by gluing together torus orbits. Each cone of Σ determines an affine toric variety, and the fan tells us how to glue them together. For example, for $\mathbb{P}^2$ the fan tells us to construct $\mathbb{P}^2$ by gluing together the three affine charts $\{x \in \mathbb{P}^2 : x_i \neq 0\}$ for $0 \leq i \leq 2$.

For more background on toric varieties, some good references include [Ful93] and the new book [CLS11].

The connection to tropical geometry begins with the following question.

Question 7.3. Given a toric variety X_Σ, and a subvariety $\overline{Y} \subseteq X_\Sigma$, which T-orbits of X_Σ does $\overline{Y}$ intersect?

The answer surprisingly uses tropical geometry. The subvariety $\overline{Y}$ intersects the torus orbit indexed by $\sigma \in \Sigma$ if and only if $\text{trop}(\overline{Y} \cap T)$ intersects $\text{relint}(\sigma)$. This follows from work of Tevelev [Tev07].

Example 7.4. Let $\overline{Y} = V(x + y + z) \subseteq \mathbb{P}^2$. Then $\overline{Y} \cap T) = \text{trop}(x + y + 1)$, which is the standard tropical line. This intersects every one of the fan of $\mathbb{P}^2$ except for the top-dimensional ones. Indeed, the top-dimensional cones correspond to the T-fixed points $[1 : 0 : 0], [0 : 1 : 0]$ and $[0 : 0 : 1]$, which do not lie in $\overline{Y}$, while every other T-orbit does contain a point of $\overline{Y}$.

Question 7.5. Given a subvariety $Y \subseteq T$, how can we find a good compactification of Y?

Example 7.6. Let $\mathcal{A} = \{H_1, \dots, H_s\}$ be a hyperplane arrangement in $\mathbb{P}^n$, where $H_i = \{x \in \mathbb{P}^n : a_i \cdot x = 0\}$, where $a_i \in \mathbb{C}^{n+1}$. Let $Y = \mathbb{P}^n \setminus \mathcal{A}$. This can be embedded into $(\mathbb{C}^*)^{s+1}$ by sending $y \in Y$ to $[a_1 \cdot y : \dots : a_s \cdot y]$. Then $Y = V(\sum_{i=1}^n b_{ij} x_j : 1 \leq i \leq s - n - 1\}$, where B is a $(s - n - 1) \times (n + 1)$ matrix of rank $s - n - 1$ with $AB^T = 0$, where A is the matrix with columns the vectors a_i. One choice of compactification of Y is the original $\mathbb{P}^n$; another is the DeConcini/Procesi wonderful compactification.

Definition 7.7. Fix $Y \subset T$, and choose a fan Σ with support $\operatorname{trop}(\Sigma)$. The closure $\overline{Y} = \operatorname{cl}(Y \subset X_\Sigma)$ is a tropical compactification of Y.

Tropical compactifications have nice properties:

(1) $\overline{Y}$ is proper.
(2) $\overline{Y}$ intersects a codimension-k T-orbit of X_Σ in codimension k.

If the fan Σ is chosen to be sufficiently refined then we get further niceness properties. One way to guarantee "sufficiently refined" is choose a fan Σ so that $\operatorname{in}_w(I(Y))$ is constant for all $w \in \operatorname{relint}\sigma$ for all $\sigma \in \Sigma$. By further refining Σ we can also assume that the toric variety X_Σ is smooth. In this case we have:

(1) The multiplicity w_σ on a maximal cone σ of Σ equals the intersection number $[\overline{Y}] \cdot [V(\sigma)]$, where $V(\sigma)$ is the closure of the T-orbit corresponding to σ.
(2) (with some additional niceness conditions) For any $\sigma \in \Sigma$, the intersection of $\overline{Y}$ with $\mathcal{O}_\sigma$ is $V(\operatorname{in}_w(I(Y)))/(\mathbb{C}^*)^{\dim\sigma}$ for any $w \in \operatorname{relint}(\sigma)$.

Example 7.8. When $Y = \mathbb{P}^n \setminus \mathcal{A}$, $\operatorname{trop}(Y) \subseteq \mathbb{R}^{s-1}$ has a coarsest fan structure. The tropical compactification $\overline{Y}$ coming from this is the DeConcini-Procesi wonderful compactification for most choices of $\mathcal{A}$.

A motivating example of this is given by the moduli space $\overline{M}_{0,n}$.

The moduli space $M_{0,n}$ parameterizes smooth genus zero curves with n distinct marked point. It thus parameterizes ways to arrange n distinct point on $\mathbb{P}^1$ up to $\operatorname{Aut}(\mathbb{P}^1)$. For example, $M_{0,3}$ is a point, as there is an automorphim of $\mathbb{P}^1$ that takes any three distinct points to $0, 1, \infty$. When $n = 4$, $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. In general,

$$\begin{aligned} M_{0,n} &= (\mathbb{P}^1 \setminus \{0,1,\infty\})^{n-3} \setminus \text{ diagonals} \\ &= (\mathbb{C}^* \setminus \{1\})^{n-3} \setminus \text{ diagonals} \\ &= \mathbb{P}^{n-3} \setminus \{x_0 = 0, x_i = 0, x_i = x_0, x_i = x_j : 1 \le i < j \le n\}. \end{aligned}$$

We thus have $M_{0,n}$ as the complement of $\binom{n-1}{2} = \binom{n}{2} - n + 1$ hyperplanes. This means the moduli space $M_{0,n}$ can be be embedded into $(\mathbb{C}^*)^{\binom{n}{2}-n}$ as a closed subvariety. The tropical variety $\operatorname{trop}(M_{0,n})$ is an $(n-3)$-dimensional fan Δ in $\mathbb{R}^{\binom{n}{2}-n}$. The toric variety X_Δ is smooth (but not complete).

The fan Δ is the space of phylogenetic trees. Maximal cones are labelled by trivalent trees with n labelled leaves. A point in the cone records the length of the internal edges in the tree.

A picture of Δ when $n = 5$ is shown in Figure 16. This is a two-dimensional fan in $\mathbb{R}^5$, so its intersection with the four-dimensional sphere in $\mathbb{R}^5$ is a graph, which is drawn in Figure 16.

The closure of $M_{0,n}$ in X_Δ is the moduli space $\overline{M}_{0,n}$ of *stable* genus zero curves with n marked points. This parameterizes trees of $\mathbb{P}^1$s with n marked points and at least three special points (nodes or marked points) on each component.

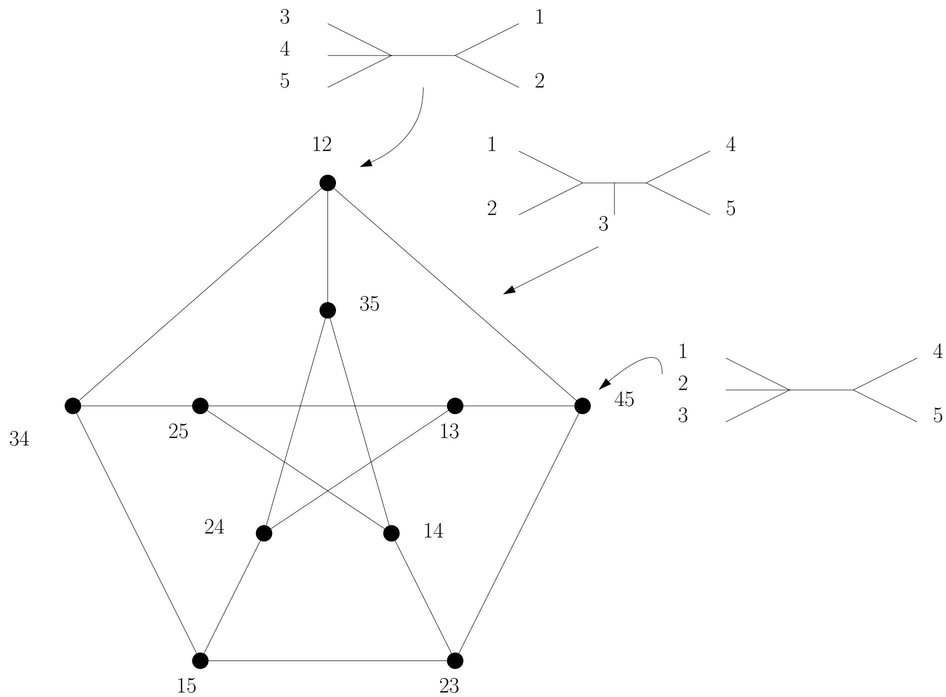


Figure 16.

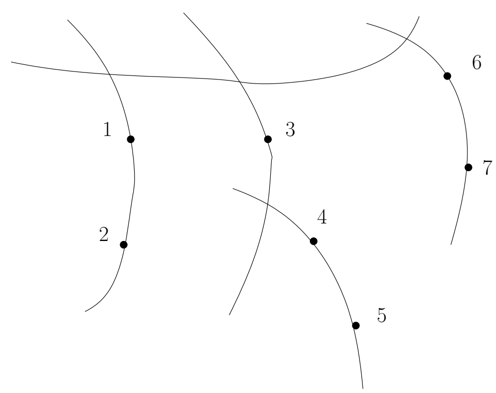


Figure 17. A stable curve with 7 marked points

For $\sigma \in \Delta$, the intersection of $\overline{M}_{0,n}$ with the torus orbit corresponding to σ is the stratum of all curves with dual graph the corresponding tree. In particular, the intersection of $\overline{M}_{0,n}$ with a torus-invariant divisor on X_Δ is a boundary divisor.

The moduli space $\overline{M}_{0,n}$ and the toric variety X_Δ are closely related. Their Picard groups are isomorphic, and the inclusion $i : \overline{M}_{0,n} \to X_\Delta$ introduces an isomorphism $i^* : A^*(X_\Delta) \to A^*(\overline{M}_{0,n})$.

8. Lecture 5

References

[CLS11] David Cox, John Little, and Hal Schenck, *Toric Varieties*, 2011.

[Ful93] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131, Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry. MR **1234037 (94g:**14028)

[Jen] Anders N. Jensen, *Gfan, a software system for Gröbner fans and tropical varieties.* Available at `http://home.imf.au.dk/jensen/software/gfan/gfan.html`.

[Mik06] Grigory Mikhalkin, *Tropical geometry and its applications*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 827–852. MR **2275625 (2008c:**14077)

[Mik] ———, *Introduction to Tropical Geometry (notes from the IMPA lectures).* arXiv:0709.1049.

[Tev07] Jenia Tevelev, *Compactifications of subvarieties of tori*, Amer. J. Math. **129** (2007), no. 4, 1087–1104, DOI 10.1353/ajm.2007.0029. MR **2343384 (2008f:**14068)