POLYHEDRAL STRUCTURES ON TROPICAL VARIETIES

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1. Introduction

One reason for the recent success of tropical geometry is that tropical varieties are easier to understand than classical varieties. This is largely because they are discrete, combinatorial objects having the structure of a polyhedral complex. The purpose of these expository notes is to give the Gröbner perspective on the origin of this polyhedral complex structure.

We begin by setting notation. Throughout we work with a fixed field K with a nontrivial valuation $\mathrm{val} : K^* \to \mathbb{R}$. We denote by R the valuation ring of K: $R = \{a \in K : \mathrm{val}(a) \geq 0\}$. The ring R is a local ring with maximal ideal $\mathfrak{m} = \{a \in K : \mathrm{val}(a) > 0\}$ and residue field $\Bbbk = R/\mathfrak{m}$. For $a \in R$ we denote by $\overline{a}$ the image of a in $\Bbbk$. We denote by $\Gamma \subseteq \mathbb{R}$ the image of the valuation val. If $\Gamma \neq \{0\}$ then we assume $1 \in \Gamma$; this can be guaranteed by replacing val by a positive multiple.

We do not assume that K is complete, but will sometimes require that it be algebraically closed. Given an ideal over a field K without a nontrivial valuation (for example, $K = \mathbb{C}$), we can extend scalars to work over the field of generalized power series with coefficients in K.

Definition 1.1. For $f = \sum_{u \in \mathbb{Z}^n} c_u x^u \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ the set $\mathrm{trop}(V(f))$ is the non-linear locus of the piecewise linear function $\mathrm{trop}(f)$ given by $\mathrm{trop}(f)(w) = \min(\mathrm{val}(c_u) + w \cdot u)$. Let $X \subseteq T^n \cong (K^*)^n$. The tropical variety is

$$\mathrm{trop}(X) = \bigcap_{f \in I(X)} \mathrm{trop}(V(f)),$$

where $I(X) \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ is the ideal of X.

The fundamental theorem of tropical algebraic geometry is the following:

Theorem 1.2. *For a variety $X \subseteq T^n \cong (K^*)^n$, where $K = \overline{K}$, the set $\mathrm{trop}(X)$ equals the closure in the Euclidean topology on $\mathbb{R}^n$ of the set*

$$\mathrm{val}(X) = \{(\mathrm{val}(x_1), \dots, \mathrm{val}(x_n)) : x = (x_1, \dots, x_n) \in X\}.$$

See, for example, [MS, Section 3.2] for a proof. Theorem 1.2 says that the tropical variety $\mathrm{trop}(X)$ may be regarded as a "combinatorial shadow" of the variety X. We now describe a third, Gröbner, way to understand the tropical variety.

We assume now that there exists a splitting of the valuation. This is a group homomorphism $\Gamma \to K^*$ sending $w \in \Gamma$ to $t^w \in K^*$ with $\mathrm{val}(t^w) = w$. If K is the field of Puiseux series $\mathbb{C}\{\{t\}\}$ with coefficients in $\mathbb{C}$, we may take the splitting that sends $w \in \mathbb{Q}$ to $t^w \in \mathbb{C}\{\{t\}\}$. If $K = \mathbb{Q}_p$, we may take the splitting that sends $w \in \mathbb{Z}$ to p^w. If K is algebraically closed, then such a splitting always exists; see [MS, Lemma 2.1.13].

Definition 1.3. Fix $w \in \Gamma^n$. For a polynomial $f = \sum_{u \in \mathbb{Z}^n} c_u x^u \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, let $W = \operatorname{trop}(f)(w) = \min(\operatorname{val}(c_u) + w \cdot u)$. We set

$$\begin{aligned}\operatorname{in}_w(f) &= \overline{t^{-W} f(t^{w_1}x_1, \dots, t^{w_n}x_n)} \\ &= \sum_{u \in \mathbb{Z}^n} \overline{t^{w \cdot u - W} c_u} x^u \\ &= \sum_{\operatorname{val}(c_u) + w \cdot u = W} \overline{t^{-\operatorname{val}(c_u)} c_u} x^u \in \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}].\end{aligned}$$

Example 1.4. Let $f = 6x^2 + 5xy + 7y^2 \in \mathbb{Q}[x^{\pm 1}, y^{\pm 1}]$, where val is the 2-adic valuation on $\mathbb{Q}$. For $w = (1, 2)$, we have $W = \min(3, 3, 4) = 3$, so

$$\begin{aligned}\operatorname{in}_w(f) &= \overline{1/8(6(2x)^2 + 5(2x)(4y) + 7(4y)^2)} \\ &= \overline{3x^2 + 5xy + 14y^2} \\ &= x^2 + xy \in \mathbb{Z}/2\mathbb{Z}[x^{\pm 1}, y^{\pm 1}].\end{aligned}$$

Definition 1.5. Let I be an ideal in $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The initial ideal of I is

$$\operatorname{in}_w(I) = \langle \operatorname{in}_w(f) : f \in I \rangle \subseteq \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}].$$

A subset $\{g_1, \dots, g_r\}$ of I is a Gröbner basis for I with respect to w if $\operatorname{in}_w(I) = \langle \operatorname{in}_w(g_1), \dots, \operatorname{in}_w(g_r) \rangle$.

This generalizes the notion of Gröbner bases for ideals in a polynomial ring with no valuations considered. An excellent elementary reference for that case is [CLO07]. As in that situation, a generating set for I need not be a Gröbner basis.

Example 1.6. Let $I = \langle x + 2y, x + 4z \rangle \subseteq \mathbb{Q}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$, where $\mathbb{Q}$ has the 2-adic valuation. For $w = (1, 1, 1)$, we have $\operatorname{in}_w(I) = \langle x, y \rangle \subseteq \mathbb{Z}/2\mathbb{Z}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$, even though $\operatorname{in}_w(x + 2y) = \operatorname{in}_w(x + 4z) = x$.

Remark 1.7. For $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the non-linear locus of the function $\operatorname{trop}(f)$ is the locus where the minimum is achieved at least twice, and thus is the closure of the collection of w for which $\operatorname{in}_w(f)$ is not a monomial. This means that, if the valuation on K is nontrivial, $\operatorname{trop}(X)$ is the closure of those $w \in \Gamma^n$ for which $\operatorname{in}_w(I(X)) \neq \langle 1 \rangle$.

2. Gröbner complex

In this section we develop the theory of the Gröbner complex of an ideal, which leads to a polyhedral structure on $\operatorname{trop}(X)$. We first restrict to the case that I is a homogeneous ideal in the (non-Laurent) polynomial ring $K[x_0, \dots, x_n]$. We assume here that $\Gamma = \operatorname{im} \operatorname{val}$ is a dense subset of $\mathbb{R}$ containing $\mathbb{Q}$. This follows from the assumption that $1 \in \Gamma$ if K is algebraically closed. If I is defined over a field with a trivial valuation, choose K to be any extension field with a nontrivial valuation, and consider $I \otimes K$; the results do not depend on the choice of K. For $w \in \Gamma^{n+1}$, the initial form $\operatorname{in}_w(f)$ of a polynomial $f \in K[x_0, \dots, x_n]$ is defined as in the Laurent polynomial case: $\operatorname{in}_w(f) = \overline{t^{-\operatorname{trop}(f)(w)} f(t^{w_1}x_1, \dots, t^{w_n}x_n)}$. The initial ideal of an ideal is similarly the ideal generated by all initial forms of polynomials in the ideal.

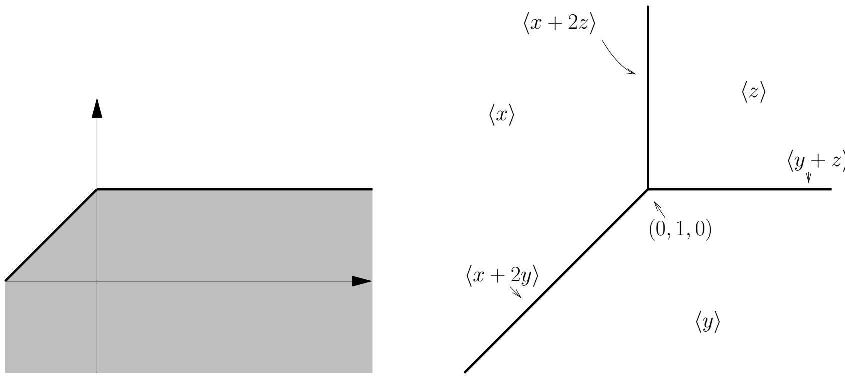

FIGURE 1.

Definition 2.1. Fix $w \in \Gamma^{n+1}$. Define

$$C_I[w] = \{w' \in \Gamma^{n+1} : \mathrm{in}_{w'}(I) = \mathrm{in}_w(I)\}.$$

We denote by $\overline{C_I[w]}$ the closure of $C_I[w]$ in the usual Euclidean topology on $\mathbb{R}^{n+1}$.

Example 2.2. Let $f = 3x + 8y + 6z \in \mathbb{Q}[x, y, z]$, where $\mathbb{Q}$ has the 3-adic valuation, and let $I = \langle f\rangle$. Fix $w = (1, 1, 1)$. Then $\mathrm{trop}(f)(w) = \min(2, 1, 2) = 1$, so $\mathrm{in}_w(f) = \overline{1/3(9x + 24y + 18z)} = 2y \in \mathbb{Z}/3\mathbb{Z}[x, y, z]$. Then

$$\begin{aligned} C_I[w] &= \{w' \in \Gamma^3 : \mathrm{in}_{w'}(I) = \langle y\rangle\} \\ &= \{w' \in \Gamma^3 : w'_1 + 1 > w'_2, w'_3 + 1 > w'_2\}. \end{aligned}$$

The closure $\overline{C_I[w]}$ is then $\{w' \in \mathbb{R}^3 : w'_1 + 1 \geq w_2, w'_3 + 1 \geq w'_2\}$. To visualize this, we note that if $w' \in \overline{C_I[w]}$, then so is $w' + \lambda(1, 1, 1)$ for any $\lambda \in \mathbb{R}$, so we may quotient by the span of $(1, 1, 1)$ to draw pictures. The region $\overline{C_I[w]}$ is the shaded region on the left of Figure 1, where we have chosen the representatives for cosets in $\mathbb{R}^3/\mathbb{R}(1, 1, 1)$ with last coordinate zero.

The picture on the right of Figure 1 shows the other possible initial ideals of I, and the corresponding regions $\overline{C_I[w]}$.

Remark 2.3. Note that if I is a homogeneous ideal in $K[x_0, \dots, x_n]$, then $\mathrm{in}_{w+\lambda\mathbf{1}}(I) = \mathrm{in}_w(I)$ for any $\lambda \in \mathbb{R}$, where $\mathbf{1} = (1, \dots, 1)$.

Recall that a polyhedral complex is a collection of polyhedra which contains all faces of any polyhedron in the collection, and for which the intersection of any two polyhedra is either empty or a common face. The key result of this section, which is proved in the following section, is that there are only finitely many of the sets $\overline{C_I[w]}$ as w varies over Γ^{n+1}, and these sets are polyhedra that fit together to form a polyhedral complex.

Every polyhedron in $\mathbb{R}^{n+1}$ can be written in the form $P = \{x \in \mathbb{R}^{n+1} : Ax \leq b\}$ where A is an $s \times (n + 1)$ matrix and $b \in \mathbb{R}^s$. We say that P is Γ-rational if the entries of A are rational, and $b \in \Gamma^s$. This means that all facet normals of P are vectors in $\mathbb{Q}^{n+1}$, and all vertices of P are elements of Γ^{n+1}. A polyhedral complex Σ is Γ-rational if all polyhedra in Σ are Γ-rational.

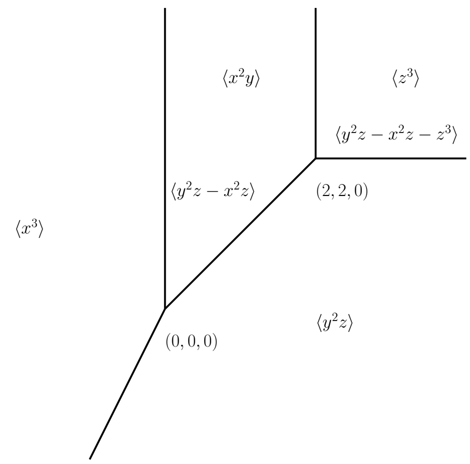

FIGURE 2.

Theorem 2.4. *Fix a homogeneous ideal $I \subseteq K[x_0, \dots, x_n]$. Then $\{\overline{C_I[w]} : w \in \Gamma^{n+1}\}$ forms a finite Γ-rational polyhedral complex.*

The polyhedral complex of Theorem 2.4 is called the *Gröbner complex*. In the case that the residue field $\Bbbk$ is a subfield of K, and I is defined over $\Bbbk$ (such as when $I \subseteq \mathbb{C}[x_0, \dots, x_n]$, where it is standard to take $K = \mathbb{C}\{\{t\}\}$), the Gröbner complex is a rational polyhedral fan, which is known as the Gröbner fan. This is well studied in the usual Gröbner literature; see [MR88] or [BM88] for the original works, or [Stu96, Chapter 2] or [MT07, Chapter 2] for expositions.

The lineality space of a polyhedral complex Σ is the largest subspace L for which if $u \in \sigma$ for any $\sigma \in \Sigma$, and $l \in L$, then $u + l \in \sigma$. Remark 2.3 thus says that $\mathbb{R}\mathbf{1}$ is in the lineality space of the Gröbner complex. The support of a polyhedral complex $\Sigma \subseteq \mathbb{R}^{n+1}$ is the collection of vectors $w \in \mathbb{R}^{n+1}$ with $w \in \sigma$ for some $\sigma \in \Sigma$.

Example 2.5. Let $I = \langle y^2z - x^3 - x^2z - p^4z^3 \rangle \subseteq \mathbb{Q}[x, y, z]$, where $\mathbb{Q}$ has the p-adic valuation for some prime p. For $f = y^2z - x^3 - x^2z - p^4z^3$, we have $\operatorname{trop}(f) = \min(2y + z, 3x, 2x + z, 3z + 4)$. The Gröbner complex is illustrated in Figure 2.

The relevance of Theorem 2.4 in the tropical context is that it gives the structure of a polyhedral complex to $\operatorname{trop}(X)$.

Given an ideal $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, we denote by $I^h \in K[x_0, \dots, x_n]$ the homogenization of $I \cap K[x_1, \dots, x_n]$. This is the ideal $I^h = \langle \tilde{f} : f \in I \cap K[x_1, \dots, x_n] \rangle$, where $\tilde{f} = x_0^{\deg(f)} f(x_1/x_0, \dots, x_n/x_0)$ is the homogenization of f.

Corollary 2.6. *Let X be a subvariety of T^n. Then there is a finite Γ-rational polyhedral complex Σ whose support $|\Sigma|$ equals* $\operatorname{trop}(X)$.

Proof. Let $I = I(X) \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the ideal of polynomials vanishing on X, and let I^h be its homogenization. It is straightforward to check that for $w \in \Gamma^n$ we have $\operatorname{in}_{(0,w)}(I^h)|_{x_0=1} = \operatorname{in}_w(I)$, where the equality is as ideals in $\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$; see [MS, Proposition 2.5.1] for details. Thus $\operatorname{in}_w(I) = \langle 1 \rangle$ if and only if $\operatorname{in}_{(0,w)}(I^h) \subseteq \Bbbk[x_0, \dots, x_n]$ contains a monomial. Let Σ be the subset of the Gröbner complex defined by $\{\overline{C_{I^h}[(0,w)]} : \operatorname{in}_{(0,w)}(I^h)$ does not contain a monomial$\}$. This is a subset of a Γ-rational polyhedral complex, so the slice $w_0 = 0$ is also a Γ-rational polyhedral complex. Since the polyhedra in Σ intersect correctly, to show that $\Sigma \cap \{w_0 = 0\} = \mathrm{trop}(X)$, it only remains to check that if $w' \in \overline{C_{I^h}[(0,w)]} \setminus C_{I^h}[(0,w)]$, then $\mathrm{in}_{w'}(I^h)$ also contains a monomial. This follows from Corollary 3.4, as if $w' \in \overline{C_{I^h}[(0,w)]}$ then there is $\mathbf{v} \in \Gamma^n$ for which $w' + \epsilon\mathbf{v} \in C_{I^h}[(0,w)]$ for all ϵ sufficiently small. Thus $\mathrm{in}_w(I^h)$ is an initial ideal of $\mathrm{in}_{w'}(I^h)$, so if $\mathrm{in}_{w'}(I^h)$ contains a monomial then so does $\mathrm{in}_w(I^h)$. Thus if $\overline{C_{I^h}[(0,w)]} \in \Sigma$, we also have $\overline{C_{I^h}[w']} \in \Sigma$ as required.

□

A drawback of the definition of a tropical variety given in Definition 1.1 is that a priori it requires taking the intersection over infinitely many tropical hypersurfaces $\mathrm{trop}(V(f))$. A second tropical consequence of Theorem 2.4 is that this infinite intersection is in fact a finite intersection.

Definition 2.7. Let $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be an ideal. A collection $\{f_1, \dots, f_r\} \subseteq I$ is a *tropical basis* for I if

$$\mathrm{trop}(V(I)) = \bigcap_{i=1}^{r} \mathrm{trop}(V(f_i)),$$

and $I = \langle f_1, \dots, f_r \rangle$.

Theorem 2.8. *Let $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be an ideal. Then a tropical basis for I always exists.*

Proof. The Gröbner complex $\Sigma(I)$ of I^h is a polyhedral complex in $\mathbb{R}^{n+1}$ with lineality space containing $\mathbb{R}\mathbf{1}$. For each of the finitely many polyhedra $\sigma^{(i)}$ in that complex, we select one representive vector $(0, w^{(i)}) \in \Gamma^{n+1}$. For each index i such that $\mathrm{in}_{w^{(i)}}(I^h)$ contains a monomial we we select a polynomial $f^{(i)} \in I^h$ such that $\mathrm{in}_{w^{(i)}}(f^{(i)})$ is a monomial x^{u_i}. Choose $\mathbf{v}_i$ with $\mathrm{in}_{\mathbf{v}_i}(\mathrm{in}_{w^{(i)}}(I^h))$ a monomial ideal; this is possible by Lemma 3.2. By Corollary 3.4 we can find $\epsilon > 0$ such that $\mathrm{in}_{\mathbf{v}_i}(\mathrm{in}_{w^{(i)}}(I^h)) = \mathrm{in}_{w^{(i)}+\epsilon\mathbf{v}_i}(I^h)$. By Lemma 3.3 there is a polynomial $g^{(i)} \in I$ of the form $x^{u_i} + \sum c_{ai} x^a$, where $c_{ai} \neq 0$ implies that $x^a \notin \mathrm{in}_{\mathbf{v}_i}(\mathrm{in}_{w^{(i)}}(I^h))$. Then for every $w \in \Gamma^{n+1}$ with $\mathrm{in}_w(I^h) = \mathrm{in}_{w^{(i)}}(I^h)$ we claim that $\mathrm{in}_w(g^{(i)}) = x^{u_i}$. Indeed, $\mathrm{in}_{\mathbf{v}_i}(\mathrm{in}_w(g^{(i)})) \in \mathrm{in}_{\mathbf{v}_i}(\mathrm{in}_{w^{(i)}}(I^h)$, and every monomial occuring in this polynomial must occur in $g^{(i)}$, but also be in the monomial ideal $\mathrm{in}_{\mathbf{v}_i}(\mathrm{in}_{w^{(i)}}(I^h))$, so must be x^{u_i}. Thus $\mathrm{in}_w(g^{(i)}) = x^{u_i} + \sum b_a x^a$ where $x^a \notin \mathrm{in}_{\mathbf{v}_i}(\mathrm{in}_w(I^h))$. Since $x^{u_i} \in \mathrm{in}_w(I^h)$, this means that $\sum b_a x^a \in \mathrm{in}_w(I^h)$, and thus $\mathrm{in}_{\mathbf{v}_i}(\sum b_a x^a) \in \mathrm{in}_{\mathbf{v}_i}(\mathrm{in}_w(I^h))$, which would contradict $x^a \notin \mathrm{in}_{\mathbf{v}_i}(\mathrm{in}_w(I^h))$ unless $\sum b_a x^a = 0$. Thus $\mathrm{in}_w(g^{(i)}) = x^{u_i}$.

Now we define a tropical basis $\mathcal{T}$ by taking any finite generating set of I and augmenting it by the polynomials $g^{(i)}|_{x_0=1}$ where $g^{(i)}$ is as constructed above. Then $\mathcal{T}$ is a generating set of I. The intersection $\bigcap_{f \in \mathcal{T}} \mathrm{trop}(V(f))$ contains $\mathrm{trop}(V(I))$ by the definition of $\mathrm{trop}(V(I))$. Consider an arbitrary weight vector $w \in \Gamma^n \setminus \mathrm{trop}(V(I))$. There exists an index i such that $\mathrm{in}_{(0,w)}(I^h) = \mathrm{in}_{w^{(i)}}(I^h)$, and this initial ideal must contain a monomial since $w \notin \mathrm{trop}(V(I))$. Thus the above argument shows that $\mathrm{in}_{(0,w)}(g^{(i)}) = x^{u_i}$, so $w \notin \mathrm{trop}(V(g^{(i)}))$. Thus $w \notin \bigcap_{f \in \mathcal{T}} \mathrm{trop}(V(f))$ and so $\mathcal{T}$ is a finite tropical basis as required. □

Remark 2.9. Hept and Theobald show in [HT09] that if $X \subseteq T^n$ is an irreducible d-dimensional variety, then there always exist $f_0, \dots, f_{n-d} \in I(X)$ with $\operatorname{trop}(X) = \bigcap_{i=0}^{n-d} \operatorname{trop}(V(f_i))$. This means that if we drop the ideal generation requirement then a tropical basis with $n - d + 1$ elements always exists. Note, however, that the degrees of the f_i may be very large. There are classical complete intersections which are not the intersection of the tropicalizations of any generating set of cardinality the codimension.

Alessandrini and Nesci give in [AN09] a uniform bound on the degrees of polynomials f_i in a tropical basis for an ideal I that depends only on the Hilbert polynomial of a homogenization of I. Thus we can bound either the size, or the degrees, of elements of a tropical basis. However at the time of writing a truly effective and efficient algorithm to compute tropical bases does not exist.

Remark 2.10. We warn that the polyhedral complex structure constructed here on $\operatorname{trop}(X)$ is not canonical, but depends on the choice of embedding of T^n into $\mathbb{P}^n$ (or, algebraically, on the choice of coordinates for the Laurent polynomial ring). As an explicit example of this phenomenon, let $I = \langle a + b + c + d + e, 3b + 5c + 7d + 11e\rangle \subseteq \mathbb{C}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}, e^{\pm 1}]$, and consider the plane $X = V(I) \subseteq (\mathbb{C}^*)^5$. The Gröbner fan of I has a one-dimensional lineality space, spanned by $(1, 1, 1, 1, 1)$. Modulo the lineality space, the Gröbner fan structure on the tropical variety of X five rays, and ten two-dimensional cones, which are the span any two of the rays. Let $\phi^* : \mathbb{C}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}, e^{\pm 1}] \to \mathbb{C}[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}, d^{\pm 1}, e^{\pm 1}]$ be the automorphism given by $\phi^*(a) = ab$, $\phi^*(b) = bc$, $\phi^*(c) = cd$, $\phi^*(d) = de$ and $\phi^*(e) = e$, and let $\phi : (\mathbb{C}^*)^5 \to (\mathbb{C}^*)^5$ be the corresponding morphism. Let $Y = \phi(X) = V(\phi^{*-1}(I))$. The set $\operatorname{trop}(Y)$ is the image of $\operatorname{trop}(X)$ under the change of coordinates given by $\operatorname{trop}(\phi^{-1})$, but the Gröbner fan structure on $\operatorname{trop}(Y)$ has seven rays and twelve cones, as two of the two-dimensional cones are subdivided. This can be verified using the software `gfan` [Jen].

A possible objection to this example is that the polyhedral structure on $\operatorname{trop}(Y)$ refines the polyhedral structure on $\operatorname{trop}(X)$, so that is a more natural polyhedral structure. However such a coarsest polyhedral structure does not always exist; see [ST08, Example 5.2].

Remark 2.11. Our construction of initial ideals depended on the choice of a splitting $w \mapsto t^w$ of the valuation map $\operatorname{val} : K^* \to \mathbb{R}$. This is necessary to be able to compare initial ideals with respect to different choices of w, as this choice makes our initial ideals into ideals in $\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ or $\Bbbk[x_0, \dots, x_n]$.

The more invariant choice recognizes that the Laurent polynomial ring is the group ring $K[M]$, where M is a lattice with dual lattice $N = \operatorname{Hom}(M, \mathbb{Z})$, and $\operatorname{val}(X)$ more naturally lives in $N \otimes \mathbb{R}$, since $T^n \cong N \otimes K^*$. We then consider the tilted group ring $R[M]^w = \{f = \sum c_u x^u : \operatorname{val}(c_u) + w \cdot u \geq 0\}$, which contains the ideal $\mathfrak{m} = \{f = \sum c_u x^u \in R[M]^w : \operatorname{val}(c_u) + w \cdot u > 0\}$. We can then define $\operatorname{in}_w(I) = (I \cap R[M]^w) + \mathfrak{m} \in R[M]^w/\mathfrak{m}$. See [Pay09] for this approach.

We note, though, that the choice of of splitting is not a very serious one. Suppose $\phi_1, \phi_2 : \Gamma \to K^*$ are two different splittings of val, so $\operatorname{val} \circ \phi_1 = \operatorname{val} \circ \phi_2 = \mathrm{id} : \Gamma \to \Gamma$. These homomorphisms induce isomorphisms $\phi_j : K[M] \to K[M]$ by $x_i \mapsto \phi_j(w_i) x_i$ for $j = 1, 2$, which restrict to isomorphisms $\phi_j : R[M]^w \to R[M]$ as we have

$\phi_j(\sum c_u x^u) = \sum c_u \phi_j(w \cdot u) x^u$, so if $\mathrm{val}(c_u) + w \cdot u \geq 0$, we have $\mathrm{val}(c_u \phi_j(w \cdot u)) \geq 0$. Thus $\psi = \phi_1 \circ \phi_2^{-1} : R[M] \to R[M]$ is an automorphism. Since ψ is the restriction of the automorphism of $K[M]$ given by $x_i \mapsto \phi_1(w_i)/\phi_2(w_i) x_i$, ψ maps the ideal $\mathfrak{m}$ to itself, so induces an automorphism $\overline{\psi} : \Bbbk[M] \to \Bbbk[M]$.

This means that the two initial ideals of I with respect to w obtained using the splittings ϕ_1 and ϕ_2 are related by the automorphism $\overline{\psi}$, so all invariants of the initial ideal such as dimension are independent of the choice of splitting. We also emphasize that such a choice is necessary to do computations. One could view (parts of) tropical geometry as the computational arm of rigid analytic geometry and Berkovich theory, so it is important not to ignore the computational aspects.

3. Proofs

This section contains the technical details needed to prove Theorem 2.4.

Lemma 3.1. *For all $f \in K[x_0, \dots, x_n]$ there is $\epsilon > 0$ such that $\mathrm{in}_v(\mathrm{in}_w(f)) = \mathrm{in}_{w+\epsilon' v}(f)$ for all $\epsilon' \in \Gamma$ with $0 < \epsilon' < \epsilon$.*

Proof. Let $f = \sum_{u \in \mathbb{N}^{n+1}} c_u x^u$. Then $\mathrm{in}_w(f) = \sum_{u \in \mathbb{N}^{n+1}} \overline{c_u t^{w \cdot u - W}} x^u$, where $W = \mathrm{trop}(f)(w)$. Let $W' = \min(v \cdot u : \mathrm{val}(c_u) + w \cdot u = W)$. Then

$$\mathrm{in}_v(\mathrm{in}_w(f)) = \sum_{v \cdot u = W'} \overline{c_u t^{w \cdot u - W}} x^u.$$

For all sufficiently small $\epsilon > 0$, we have $W + \epsilon W' = \mathrm{trop}(f)(w + \epsilon v)$ and

$$\{u : \mathrm{val}(c_u) + (w + \epsilon' v) \cdot u = W + \epsilon W'\} = \{u : \mathrm{val}(c_u) + w \cdot u = W, v \cdot u = W'\}.$$

This implies $\mathrm{in}_{w+\epsilon' v}(f) = \mathrm{in}_v(\mathrm{in}_w(f))$ for all $\epsilon' \in \Gamma$ with $0 < \epsilon' < \epsilon$. $\square$

In standard Gröbner bases most attention is paid to initial ideals that have a monomial generating set. Such monomial ideals are useful because their properties only depend on the set of monomials in the ideal. For example, a polynomial $f = \sum c_u x^u$ lies in a monomial ideal if and only if every x^u with $c_u \neq 0$ lies in the ideal. We next check that in this modified Gröbner theory monomial initial ideals still exist.

Lemma 3.2. *Let I be a homogeneous ideal in $K[x_0, \dots, x_n]$, and fix $w \in \Gamma^{n+1}$. Then there is $\mathbf{v} \in \mathbb{Q}^{n+1}$ and $\epsilon > 0$ for which both $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I))$ and $\mathrm{in}_{w+\epsilon\mathbf{v}}(I)_d$ are monomial ideals, and $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I)) \subseteq \mathrm{in}_{w+\epsilon\mathbf{v}}(I)$.*

Note that in Corollary 3.4 we will show that for sufficiently small $\epsilon > 0$ these two initial ideals are equal.

Proof. Given any $\mathbf{v} \in \mathbb{Q}^{n+1}$, let $M_{\mathbf{v}}$ denote the ideal generated by all monomials in $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I))$, and let $M_{\mathbf{v}}^{\epsilon}$ denote the ideal generated by all monomials in $\mathrm{in}_{w+\epsilon\mathbf{v}}$ for some $\epsilon > 0$. Choose $\mathbf{v} \in \mathbb{Q}^{n+1}$ for which there is no $\mathbf{v}' \in \mathbb{Q}^{n+1}$ with $M_{\mathbf{v}} \subsetneq M_{\mathbf{v}'}$, which is possible since the polynomial ring is Noetherian. If $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I))$ is not a monomial ideal, then there is $f \in I$ with none of the terms of $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(f))$ lying in $M_{\mathbf{v}}$. Choose $\mathbf{v}' \in \mathbb{Q}^{n+1}$ with $\mathrm{in}_{\mathbf{v}'}(\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(f)))$ a monomial; any $\mathbf{v}'$ for which the face of the Newton polytope of $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(f))$ is a vertex suffices. By Lemma 3.1 there is $\epsilon' > 0$ for which $\mathrm{in}_{\mathbf{v}+\epsilon'\mathbf{v}'}(\mathrm{in}_w(f))$ is this monomial. By choosing ϵ' sufficiently small we can guarantee that $\mathrm{in}_{\mathbf{v}+\epsilon'\mathbf{v}'}(\mathrm{in}_w(I))_d$ contains all generators of $M_{\mathbf{v}}$, as a generator x^u is $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(f))$

for some $f \in I$ so this follows from applying Lemma 3.1 to $\mathrm{in}_w(f)$. Thus contradicts the choice of $\mathbf{v}$, so we conclude that $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I)) = M_{\mathbf{v}}$.

Choose $f_1, \dots, f_s$ for which $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(f_i)) = x^{u_i}$, where the x^{u_i} generate $M_{\mathbf{v}}$. By Lemma 3.1 there is $\epsilon > 0$ for which $\mathrm{in}_{w+\epsilon\mathbf{v}}(f_i) = x^{u_i}$ for all i, so for this ϵ we have $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I)) \subseteq \mathrm{in}_{w+\epsilon\mathbf{v}}(I)$. Suppose that $\mathbf{v}$ has been chosen from those $\mathbf{v}$ with $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I))$ monomial to maximize $M^{\epsilon}_{\mathbf{v}}$. Again, if $\mathrm{in}_{w+\epsilon\mathbf{v}}(I)$ is not monomial then there is $f \in I$ with no term of $\mathrm{in}_{w+\epsilon\mathbf{v}}(f) \in M^{\epsilon}_{\mathbf{v}}$, and we can choose $\mathbf{v}'$ as above so that $M^{\epsilon}_{\mathbf{v}} \subsetneq M^{\epsilon}_{\mathbf{v}+\epsilon'\mathbf{v}'}$. From this contradiction we conclude that $\mathrm{in}_{w+\epsilon\mathbf{v}}(I)$ is also monomial, so we have constructed the desired $\mathbf{v} \in \mathbb{Q}^{n+1}$. $\square$

We denote by S_K the polynomial ring $K[x_0, \dots, x_n]$, and by $S_{\Bbbk}$ the polynomial ring $\Bbbk[x_0, \dots, x_n]$.

Lemma 3.3. *Let $I \subseteq K[x_0, \dots, x_n]$ be a homogeneous ideal, and let $w \in \Gamma^{n+1}$ be such that $\mathrm{in}_w(I)_d$ is the span of $\{x^u : x^u \in \mathrm{in}_w(I)_d\}$. Then the monomials not in $\mathrm{in}_w(I)$ of degree d form a K-basis for $(S/I)_d$. This implies that if $w \in \Gamma^{n+1}$ is arbitrary then the Hilbert function of I and $\mathrm{in}_w(I)$ agree:*

$$\dim_K(S_K/I)_d = \dim_{\Bbbk}(S_{\Bbbk}/\,\mathrm{in}_w(I))_d \text{ for all degrees } d.$$

Proof. Suppose first that $\mathrm{in}_w(I)_d$ is the span of $\{x^u : x^u \in \mathrm{in}_w(I)_d\}$. Let $\mathcal{B}_d$ be the set of monomials of degree d not contained in $\mathrm{in}_w(I)$. We first show that, regarded as elements of $(S/I)_d$, the set $\mathcal{B}_d$ is linearly independent. Indeed, if this set were linearly dependent there would exist $f = \sum c_u x^u \in I_d$, with $x^u \notin \mathrm{in}_w(I)$ whenever $c_u \neq 0$. But then $\mathrm{in}_w(f) \in \mathrm{in}_w(I)_d$, which would mean that every term of $\mathrm{in}_w(f)$ is in $\mathrm{in}_w(I)_d$, contradicting the construction of f. Since $|\mathcal{B}_d| = \binom{n+d}{n} - \dim_{\Bbbk} \mathrm{in}_w(I)_d$, this linear independence implies that $\dim_{\Bbbk} \mathrm{in}_w(I)_d \geq \dim_K I_d$.

For all monomials $x^u \in \mathrm{in}_w(I)_d$, choose polynomials $f_u \in I_d$ with $\mathrm{in}_w(f_u) = x^u$. We next note that the collection $\{f_u : x^u \in \mathrm{in}_w(I)_d\}$ is linearly independent. If not, there are $a_u \in K$ not all zero with $\sum a_u f_u = 0$. Write $f_u = x^u + \sum c_{uv} x^v$. Let u' minimize $\mathrm{val}(a_u) + w \cdot u$ for all $u \in \mathbb{N}^{n+1}$ with $x^u \in \mathrm{in}_w(I)_d$. Then $a_{u'} + \sum_{u \neq u'} a_u c_{uu'} = 0$, so there is $u'' \neq u'$ with $\mathrm{val}(a_{u''}) + \mathrm{val}(c_{u''u'}) \leq \mathrm{val}(a_{u'})$. But then $\mathrm{val}(a_{u''}) + \mathrm{val}(c_{u''u'}) + w \cdot u' \leq \mathrm{val}(a_{u'}) + w \cdot u' \leq \mathrm{val}(a_{u''}) + w \cdot u''$, which contradicts $\mathrm{in}_w(f_{u''}) = x^{u''}$. This shows $\dim_K I_d \geq \dim_{\Bbbk} \mathrm{in}_w(I)_d$. This means that when $\mathrm{in}_w(I)$ is a monomial ideal we have $\dim_K(S_K/I)_d = \dim_{\Bbbk}(S_{\Bbbk}/\,\mathrm{in}_w(I))_d$, and $\mathcal{B}_d$ is a K-basis for $(S_K/I)_d$.

If $\mathrm{in}_w(I)_d$ is not spanned by the monomials it contains, by Lemma 3.2 there is $\mathbf{v} \in \mathbb{Q}^{n+1}$ and $\epsilon > 0$ for which both $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I))_d$ and $\mathrm{in}_{w+\epsilon\mathbf{v}}(I)_d$ are spanned by the monomials they contain.

By the previous calculation the monomials not in $\mathrm{in}_{w+\epsilon\mathbf{v}}(I)_d$ span $(S/I)_d$, so if $x^u \in \mathrm{in}_{w+\epsilon\mathbf{v}}(I)_d \setminus \mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I))_d$ there is $f_u \in I_d$ with $f_u = x^u + \sum c_v x^v$ with $c_v \neq 0$ implying that $x^v \notin \mathrm{in}_{w+\epsilon\mathbf{v}}(I)_d$. But then $\mathrm{in}_w(f_u)$ is supported on monomials not in $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I))_d$, so $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(f_u)) \notin \mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I))_d$. From this contradiction we conclude that $\mathrm{in}_{w+\epsilon\mathbf{v}}(I)_d = \mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I))_d$.

Again, by the previous calculation we have $\dim_{\Bbbk}(S_{\Bbbk}/\,\mathrm{in}_w(I))_d = \dim_{\Bbbk}(S_{\Bbbk}/\,\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I)))_d$, and $\dim_K(S_K/I)_d = \dim_{\Bbbk}(S_{\Bbbk}/\,\mathrm{in}_{w+\epsilon\mathbf{v}}(I))_d$, so we conclude that for any $w \in \Gamma^{n+1}$ we have $\dim_K(S_K/I)_d = \dim_{\Bbbk}(S_{\Bbbk}/\,\mathrm{in}_w(I))_d$ for all degrees d.

$\square$

Corollary 3.4. *Let I be a homogeneous ideal in $K[x_0,\dots,x_n]$, and let $w,\mathbf{v}\in\Gamma^{n+1}$. Then there is $\epsilon>0$ such that for all $0<\epsilon'<\epsilon$ with $\epsilon'\in\Gamma^{n+1}$ we have*

$$\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I))=\mathrm{in}_{w+\epsilon\mathbf{v}}(I).$$

Proof. Let $\{g_1,\dots,g_s\}\subset\Bbbk[x_0,\dots,x_n]$ be a generating set for $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I))$, with each generator g_i of the form $\mathrm{in}_v(\mathrm{in}_w(f_i))$ for some $f_i\in I$. We choose ϵ to be the minimum of the ϵ_i from Lemma 3.1. Then $g_i=\mathrm{in}_v(\mathrm{in}_w(f_i))=\mathrm{in}_{w+\epsilon' v}(f_i)$, so $\mathrm{in}_v(\mathrm{in}_w(I))\subseteq\mathrm{in}_{w+\epsilon' v}(I)$, for any $\epsilon'<\epsilon$. But by Lemma 3.3 both $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I))$ and $\mathrm{in}_{w+\epsilon'\mathbf{v}}(I)$ have the same Hilbert function as I, so this containment cannot be proper. □

Proposition 3.5. *Let I be a homogeneous ideal in $K[x_0,\dots,x_n]$. There are only a finite number of different monomial initial ideals $\mathrm{in}_w(I)$ as w varies over Γ^{n+1}.*

Proof. If this were not the case, by [Mac01, Theorem 1.1] there would be $w_1,w_2\in\Gamma^{n+1}$ with $\mathrm{in}_{w_2}(I)\subsetneq\mathrm{in}_{w_1}(I)$, where both initial ideals are monomial ideals. Fix $x^u\in\mathrm{in}_{w_1}(I)\setminus\mathrm{in}_{w_2}(I)$. By Lemma 3.3 the monomials of degree $\deg(x^u)$ not in $\mathrm{in}_{w_1}(I)$ form a K-basis for S/I, so there is $f_u\in I$ with $f_u=x^u+\sum c_vx^v$ where whenever $c_v\neq 0$ we have $x^v\notin\mathrm{in}_{w_1}(I)$. But then $\mathrm{in}_{w_2}(f_u)\in\mathrm{in}_{w_2}(I)$, and since $\mathrm{in}_{w_2}(f_u)$ is a monomial ideal this means that all of its terms lie in $\mathrm{in}_{w_2}(I)$. However all monomials appearing in $\mathrm{in}_{w_2}(f_u)$ appear in f_u, so this is a contradiction, and thus there are only a finite number of monomial initial ideals of I. □

Fix a homogeneous ideal $I\subseteq K[x_0,\dots,x_n]$. Proposition 3.5 guarantees that there are only finitely many different monomial initial ideals. Let D be the maximum degree of a minimal generator of any monomial initial ideal of I.

For any fixed degree d let $s=\dim_K(I_d)$, and choose a basis $f_1,\dots,f_s$ for I_d. Let A_d be the corresponding $s\times\binom{n+d}{n}$ matrix recording the coefficients of the polynomials f_i. This matrix has columns indexed by the monomials $\mathcal{M}_d$ in $K[x_0,\dots,x_n]$ of degree d, so $(A_d)_{iu}$ is the coefficient of x^u in f_i. Note that the maximal minors of this matrix are independent of the choice $f_1,\dots,f_s$ of basis, as they are the Plücker coordinates of the element I_d in the Grassmannian $\mathrm{Gr}(s,S_d)$. For $J\subseteq\mathcal{M}_d$ with $|J|=s$, we denote by A_d^J the $s\times s$ minor of A_d indexed by columns labelled by those monomials in J.

Let $g_d\in K[x_0,\dots,x_n]$ be given by

$$g_d=\sum_{I\subseteq\mathcal{M}_d,|I|=s}\det(A_d^I)\prod_{\mathbf{u}\in I}x^u.$$

Let $g=\prod_{d=1}^D g_d$. The function $\mathrm{trop}(g):\mathbb{R}^{n+1}\to\mathbb{R}$ is piecewise-linear. Let $\Sigma_{\mathrm{trop}(g)}$ be the coarsest polyhedral complex for which $\mathrm{trop}(g)$ is linear on each polyhedron in $\Sigma_{\mathrm{trop}(g)}$. Note that $\Sigma_{\mathrm{trop}(g)}$ is a Γ-rational polyhedral complex.

Theorem 3.6. *Fix a homogeneous ideal $I\subseteq K[x_0,\dots,x_n]$, and let g_d,g and $\Sigma_{\mathrm{trop}(g)}$ be as above. Fix $w\in\Gamma^{n+1}$ in the interior of a maximal polyhedron $\sigma\in\Sigma_{\mathrm{trop}(g)}$. Then $\sigma=\overline{C_I[w]}$.*

Proof. We need to show two things: firstly, that if $w'\in\Gamma^{n+1}$ lies in the interior of σ then $\mathrm{in}_{w'}(I)=\mathrm{in}_w(I)$, and secondly that if w' does not lie in the interior of σ then $\mathrm{in}_{w'}(I)$ is not equal to $\mathrm{in}_w(I)$. Note that $\Sigma_{\mathrm{trop}(g)}$ is the common refinement of the polyhedral complexes $\Sigma_{\mathrm{trop}(g_d)}$ for $d \leq D$, where $\Sigma_{\mathrm{trop}(g_d)}$ is the coarsest polyhedral complex for which $\mathrm{trop}(g_d)$ is linear on each polyhedron. Thus it suffices to restrict to a fixed $d \leq D$, and let σ_d be the polyhedron of $\Sigma_{\mathrm{trop}(g_d)}$ containing σ. We then need to show that if $w' \in \Gamma^{n+1}$ lies in the interior of σ_d then $\mathrm{in}_{w'}(I)_d = \mathrm{in}_w(I)_d$ and if w' does not lie in the interior of σ_d then $\mathrm{in}_{w'}(I)_d$ is not equal to $\mathrm{in}_w(I)_d$. This suffices because $\mathrm{in}_{w'}(I) = \mathrm{in}_w(I)$ if and only if $\mathrm{in}_{w'}(I)_d = \mathrm{in}_w(I)_d$ for all $d \leq D$.

For the first of these, note that if w' lies in the interior of σ_d then the minimum in $\mathrm{trop}(g_d)$ is achieved at the same term for w and for w'. Since σ_d is a maximal polyhedron, this minimum is achieved at only one term, which we may assume is the one indexed by $J \in \mathcal{M}_d$.

Let $\tilde{A}$ be the $s \times s$ submatrix of A_d containing those columns corresponding to monomials in J, and consider the matrix $A' = \tilde{A}^{-1}A_d$. This shifts the valuations of the minors: $\mathrm{val}(A'^J) = \mathrm{val}(A_d^J) - \mathrm{val}(\det(\tilde{A}))$. This has an identity matrix in the columns indexed by J, so each row of the matrix gives a polynomial in S_d indexed by $x^u \in J'$. Let $\tilde{f}_u = x^u + \sum_{x^v \not\in J'} c_v x^v$ be the polynomial indexed by x^u. Note that the minor of A' indexed by $J_v = J \setminus \{x^u\} \cup \{x^v\}$ for $x^v \not\in J$ is c_v, up to sign, so

$$\begin{aligned}
\mathrm{val}(A'^{J_v}) + \sum_{x^{u'} \in J_v} w \cdot u' &= \mathrm{val}(A_d^{J_v}) - \mathrm{val}(\det(\tilde{A}) + \sum_{x^{u'} \in J_v} w \cdot u' \\
&> \mathrm{val}(A_d^{J}) - \mathrm{val}(\det(\tilde{A}) + \sum_{x^{u'} \in J} w \cdot u' \\
&= \mathrm{val}(A'^{J}_d) + \sum_{x^{u'} \in J_v} w \cdot u' + w \cdot u - w \cdot v \\
&= 0 + \sum_{x^{u'} \in J_v} w \cdot u' + w \cdot u - w \cdot v
\end{aligned}$$

Thus $\mathrm{val}(c_v) + w \cdot v > w \cdot u$ for any v with $x^v \not\in J$, so $\mathrm{in}_w(\tilde{f}_u) = x^u$. This means that $x^u \in \mathrm{in}_w(I)_d$, and so, since by Lemma 3.3 $\dim_{\Bbbk} \mathrm{in}_w(I)_d = s$, J is precisely the collection of monomials in $\mathrm{in}_w(I)_d$. Since $|J| = s = \dim_k \mathrm{in}_w(I)_d = \mathrm{in}_{w'}(I)_d$ we have $\mathrm{in}_w(I)_d = \mathrm{in}_{w'}(I)_d$ as required. Note that this also shows that $\mathrm{in}_w(I)$ is a monomial ideal, since in all degrees d up to the bound D on its generators $\mathrm{in}_w(I)_d$ is spanned by monomials in $\mathrm{in}_w(I)$

For the second, suppose that w' does not lie in the interior of σ_d. This means that there is some $J' \in \mathcal{M}_d$ with $\mathrm{val}(A_d^{J'}) + \sum_{u \in J'} w' \cdot u \leq \mathrm{val}(A_d^J) + \sum_{u \in J} w' \cdot u$, where J is as above. Choose $\mathbf{v} \in \mathbb{Q}^{n+1}$ with $\mathbf{v} \cdot (\sum_{u \in J} w' \cdot u - \sum_{u \in J} w' \cdot u) < 0$. Then for sufficiently small $\epsilon > 0$ we have $\mathrm{val}(A_d^{J'}) + \sum_{u \in J'} (w' + \epsilon \mathbf{v}) \cdot u < \mathrm{val}(A_d^L) + \sum_{u \in L} w' \cdot u$ for all $L \in \mathcal{M}_d$ with $L \neq J'$. Then as above again we have $\mathrm{in}_{w'+\epsilon\mathbf{v}}(I)_d = \mathrm{span}\{x^u : x^u \in J'\}$. By Corollary 3.4 we have $\mathrm{in}_{w'+\epsilon\mathbf{v}}(I) = \mathrm{in}_{\mathbf{v}}(\mathrm{in}_{w'}(I))$, so this means that $\mathrm{in}_{w'}(I)_d$ is not the span of those monomials in J, and thus $\mathrm{in}_{w'}(I)_d \neq \mathrm{in}_w(I)_d$. $\square$

Theorem 2.4 is now a straightforward corollary of Theorem 3.6.

Proof of Theorem 2.4. Theorem 3.6 states that all top-dimensional regions of the Γ-rational polyhedral complex $\Sigma_{\mathrm{trop}(f)}$ are of the form $\overline{C_I[w]}$ for some $w \in \Gamma^{n+1}$

with $\mathrm{in}_w(I)$ a monomial ideal. For any $w \in \Gamma^{n+1}$ with $\mathrm{in}_w(I)$ a monomial ideal by Corollary 3.4 we have $\mathrm{in}_{w+\epsilon\mathbf{v}}(I) = \mathrm{in}_w(I)$ for all $\mathbf{v} \in \mathbb{Q}^n$ and all sufficiently small ϵ. This means that such a $\overline{C_I[w]}$ is full-dimensional, so it must be one of the top-dimensional regions of $\Sigma_{\mathrm{trop}(f)}$, as for $w \neq w'$ the regions $C_I[w]$ and $C_I[w']$ are either disjoint or coincide. It thus remains to show that if $\mathrm{in}_w(I)$ is not a monomial ideal, then $\overline{C_I[w]}$ is a face of some $\overline{C_I[w']}$ with $\mathrm{in}_{w'}(I)$ a monomial ideal.

This follows from Corollary 3.4 and Lemma 3.2. Indeed, by Lemma 3.2 there is some $\mathbf{v} \in \mathbb{Q}^n$ with $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I))$ a monomial ideal, and by Corollary 3.4 there is $\epsilon > 0$ for which $\mathrm{in}_{w+\epsilon\mathbf{v}}(I) = \mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I))$. Let $w' = w+\epsilon\mathbf{v}$. Let $g_1, \dots, g_s$ be a Gröbner basis for I with respect to w', so $\mathrm{in}_{w'}(I) = \langle \mathrm{in}_{w'}(g_1), \dots, \mathrm{in}_{w'}(g_s)\rangle$. Write $g_i = x^{u_i} + \sum c_{iv}x^v$, where $\mathrm{in}_w(g_i) = x^{u_i}$. We may assume, as in the proof of Lemma 3.3, that $c_{iv} \neq 0$ implies that $x^v \notin \mathrm{in}_{w'}(I)$. Then the polyhedron $\overline{C_I[w']}$ has the following inequality description:

$$\overline{C_I[w']} = \{x \in \mathbb{R}^{n+1} : u_i \cdot x \leq \mathrm{val}(c_{iv}) + x \cdot v : 1 \leq i \leq s\}.$$

To see this, note that for any $\widetilde{w} \in \Gamma^{n+1}$ properly satisfying all of these inequalities we have $\mathrm{in}_{\widetilde{w}}(g_i) = x^{u_i}$ for $1 \leq i \leq n$, so $\mathrm{in}_{w'}(I) \subseteq \mathrm{in}_{\widetilde{w}}(I)$. Since these two initial ideals have the same Hilbert function by Lemma 3.3, this containment must be an inequality, so $\widetilde{w} \in C_I[w']$. If $\widetilde{w} \in \Gamma^{n+1}$ lies outside this set, there is some g_i for which $\mathrm{in}_{\widetilde{w}}(g_i)$ does not contain x^{u_i} in its support. Let $b = \widetilde{w} \cdot u_i - \min\{\widetilde{w} \cdot v : c_{vi} \neq 0\}$. By assumption $b > 0$. If $\widetilde{w} \in \overline{C_I[w']}$ then for all $\epsilon > 0$ there is $\mathbf{u}'$ with $|\mathbf{u}'| < \epsilon$ and $\widetilde{w} + \mathbf{u}' \in C_I[w']$. Choose $\epsilon > 0$ sufficiently small so that all $\mathbf{u}'$ with $|\mathbf{u}'| < \epsilon$ satisfy $\mathbf{u}' \cdot (v - u_i) < b/2$ for all v with $c_{vi} \neq 0$. Then $\mathrm{in}_{\widetilde{w}+\mathbf{u}'}(g_i) \in \mathrm{in}_{\widetilde{w}+\mathbf{u}'}(I) = \mathrm{in}_{w'}(I)$ does not contain x^{u_i} in its support. Since $\mathrm{in}_{w'}(I)$ is a monomial ideal, all terms of $\mathrm{in}_{\widetilde{w}+\mathbf{u}'}(g_i)$ must lie in $\mathrm{in}_{w'}(I)$, which is a contradiction, so such $\widetilde{w}$ do not lie in $\overline{C_I[w']}$, and thus $\overline{C_I[w']}$ has the claimed description.

The above argument says that $C_I[w]$ lies in $\overline{C_I[w']}$, so we just need to show that it is a face. Note that $\{\mathrm{in}_w(g_1), \dots, \mathrm{in}_w(g_s)\}$ is a Gröbner basis for $\mathrm{in}_w(I)$ with respect to $\mathbf{v}$. If $\widetilde{w} \in \Gamma^{n+1}$ satisfies $\mathrm{in}_{\widetilde{w}}(I) = \mathrm{in}_w(I)$, then we must have $\mathrm{in}_{\widetilde{w}}(g_i) = \mathrm{in}_w(g_i)$. If not, $\mathrm{in}_{\widetilde{w}}(g_i)$ must still have x^{u_i} in its support, or we would not have $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_{\widetilde{w}}(I)$ equal to the monomial ideal $\mathrm{in}_{w'}(I)$. But then $\mathrm{in}_{\widetilde{w}}(g_i) - \mathrm{in}_w(g_i) \in \mathrm{in}_w(I)$, and this polynomial does not contain any monomials from $\mathrm{in}_{w'}(I)$, contradicting $\mathrm{in}_{\mathbf{v}}(\mathrm{in}_w(I)) = \mathrm{in}_{w'}(I)$. Thus $\widetilde{w}$ lies in the polyhedron

$$\{x \in \mathbb{R}^{n+1} : u_i{\cdot}x \leq \mathrm{val}(c_{iv}){+}x{\cdot}v, u_i{\cdot}x = v'{\cdot}x : 1 \leq i \leq s, x^{v'} \text{ is in the support of } \mathrm{in}_w(g_i)\}.$$

On the other hand, any $\widetilde{w} \in \Gamma^{n+1}$ lying in this set has $\mathrm{in}_{\widetilde{w}}(g_i) = \mathrm{in}_w(I)$, so $\mathrm{in}_w(I) \subseteq \mathrm{in}_{\widetilde{w}}(I)$, and so by Lemma 3.3 we have equality, so $\widetilde{w} \in C_I[w]$. Since this polyhedron is the intersection of $\widetilde{C_I[w']}$ with a supporting subspace it is a face as required. □

Remark 3.7. The construction of the Gröbner complex as the linear locus of a tropical function shows that this polyhedral complex is a *regular subdivision*. This notion originates in the work of Gelfand, Kapranov, and Zelevinsky [GKZ08, Chapter 7], where such subdivisions were called coherent; see also [DLRS10, Chapter 5]. The content here is that the piecewise linear function $\mathrm{trop}(f)$ is concave.

Remark 3.8. Note that the polynomial g is homogeneous of degree $L = \sum_{d=1}^{D} \dim_K(I_d)$. Write $g = \sum c_u x^u$, where the sum is over $u \in \mathbb{N}^{n+1}$ with $|u| = L$. When K has the trivial valuation, the linear locus of $\operatorname{trop}(g)$ is the normal fan of the polytope $\operatorname{conv}(u \in \mathbb{N}^{n+1} : c_u \neq 0)$. This polytope is known as the state polytope of I, and was first described in [BM88]. The construction given above mimics this construction; see [Stu96, Chapter 2] for an exposition in this case. When K has a nontrivial valuation, the Gröbner complex agrees with the normal fan to the state polytope of I for large w, and is the dual complex to a regular subdivision of the state polytope.

Remark 3.9. When K has the trivial valuation we do not need to assume that the ideal I is homogeneous to define the Gröbner fan. In this case Anders Jensen gave an example in [Jen07] of an ideal $I \subseteq \mathbb{C}[x_1, x_2, x_3, x_4]$ for which the Gröbner fan is not a regular subdivision. However if we take $X \subset T^4$ to be the variety defined by the ideal $I\mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 4}]$, then $\operatorname{trop}(X)$ is the support of a subcomplex of this Gröbner fan, and also a the support of a subcomplex of a regular subdivision. This is not a contradiction, as the regular subdivision coming from the Gröbner fan of the homogenization can be much finer than the nonregular one.

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