MATH 216B HOMEWORK 2

SPRING 2004

(1) Recall that the stellar subdivision of a cone $\sigma = \text{pos}(v_1, \dots, v_k)$ with respect to a vector $v \in \sigma$ has rays $\{v_1, \dots, v_k, v\}$, and cones $\text{pos}(v, v_i : v_i \in \tau')$, where τ' is a face of σ not containing v. Show that this actually is a subdivsion, so these cones fit together to form a fan that covers σ.

(2) Fix $a \in \mathbb{N}$, $a > 0$, and let $A = \mathbb{C}[t, yt, xyt, xt, x^2t, \dots, x^{a+1}t]$, with $\deg(x) = \deg(y) = 0$, and $\deg(t) = 1$. Show that $\text{Proj}(A)$ is the Hirzebruch surface $\mathbb{F}_a$ (whose fans has rays $\{(1,0), (0,1), (-1,a), (0,-1)\}$ and the four two-dimensional cones forced by these rays).

(3) Describe the fan for the weighted projective space $\mathbb{P}(a_0, \dots, a_n)$ (we did the case $a_0 = 1$ in class) (Hint: See Fulton).

(4) Find a resolution of singularities for $\mathbb{P}(1, 2, 3, 4)$.

(5) Prove that every complete smooth toric surface is a blow-up of $\mathbb{F}_a$ or $\mathbb{P}^2$. You may want to follow the following outline (and see the hints in Fulton):
   (a) Let the rays of Δ be $v_0, \dots, v_{d-1}, v_d = v_0$ in counterclockwise order. Show that $v_{i+1} + v_{i-1} = a_i v_i$ for some $a_i \in \mathbb{N}$.
   (b) We want to prove that if $d \geq 5$, there is an i such that v_{i-1} and v_i generate a strongly convex polyhedral cone, and $a_i = 1$ (explain why!).
   (c) Show that if $d \geq 4$ there must be two opposite vectors in the sequence (ie $v_i = -v_j$).
   (d) If $v_i = -v_0$, and $i \geq 3$, show that $v_j = v_{j-1} + v_j$ for some $0 < j < i$.