MATH 171, SPRING 2003, HOMEWORK 1

DUE THURSDAY, APRIL 10, IN CLASS

In this class the word *show* always means "provide a complete proof for". Working with other people on the homework in encouraged, but you must write up your own version of the collaboration to hand in. Note at the end of your homework the names of anyone you worked with. If you would like to join a homework study group but do not know anyone in the class, let me know and I'll try to put you in touch with each other.

(1) We defined a set S to be countable if it is finite or if there is a bijection from $\mathbb{N}$ to S. Show that S is countable if and only if there is a surjection from $\mathbb{N}$ to S.

(2) **Definition of the real numbers**.

Note that this question has 10 parts a) – j). For full credit you need to do a)-d) and j) - for extra credit you can add e) – i) (but this would result in a long assignment!).

A set A of rational numbers is said to be a *Dedekind cut* if it satisfies the following properties:

- A is non-empty, but $A \neq \mathbb{Q}$.
- If $q \in A$ and $p < q$ for $p \in \mathbb{Q}$ then $p \in A$.
- A contains no largest rational. That is, if $p \in A$ then there exists a $q \in A$ with $p < q$.

If $q \in \mathbb{Q}$, we define the set $\overline{q} = \{p \in \mathbb{Q} : p < q\}$.

(a) Show that $\overline{q}$ is a cut.

We call such cuts *rational cuts*. The cuts which are not of the form $\overline{q}$ for some $q \in \mathbb{Q}$ will correspond to the *irrational real numbers*. The first step to this construction is to show that the set of cuts is an ordered field.

If A and B are cuts, we define

$$A + B = \{a + b : a \in A, b \in B\}.$$

(b) Show that $A + B$ is a cut.

We say a rational number q is an *upper bound* for A if $p \leq q$ for all $p \in A$. We say that q is the *least upper bound* for A (denoted $\text{lub}(A)$) if q is an upper bound for A, and if whenever $q' \in \mathbb{Q}$ is another upper bound for A, we have $q \leq q'$. If A is a cut we set

$$-A = \{q \in \mathbb{Q} : -q \text{ is an upper bound for } A, \text{ but not a least upper bound for } A\}.$$

(c) Show that if A is a cut, then $-A$ is a cut.

(d) Show that the set of cuts satisfies axioms 1) through 4) of an ordered field from page 26 of the text, using these definitions of $+$ and $-$, and $\overline{0}$ for the zero element.

If A and B are two cuts, we say $A \leq B$ if $A \subseteq B$. We call a cut A *nonnegative* if $\overline{0} \leq A$. Otherwise we call it a *negative* cut.

If A and B are two nonnegative cuts, we set

$$AB = \{ab : a \in A, b \in B, a, b \geq 0\} \cup \{c \in \mathbb{Q} : c < 0\}.$$

If A is a negative cut we set AB to be $-((-A)B)$ for any nonnegative cut B. Similarly if B is a negative cut we set $AB = -A(-B)$ for any nonnegative cut A. Finally, if A and B are both negative cuts, then $AB = (-A)(-B)$.

(e) Show that AB is a cut if A and B are nonnegative cuts.

(f) Show that the definition is well-defined if A or B are negative cuts. That is, show that if A is a negative cut then $-A$ is a nonnegative cut. Also, show that if A is a negative cut, then $-A \neq \overline{0}$.

Finally, if A is a positive (nonnegative and not zero) cut, we set

$$\begin{aligned} A^{-1} &= \{q \in \mathbb{Q} : q > 0, 1/q \text{ is an upper bound for } A, \\ &\quad 1/q \text{ is not a least upper bound for } A\} \cup \{q \in \mathbb{Q} : q \leq 0\}. \end{aligned}$$

If A is a negative cut, we set $A^{-1} = -(-A)^{-1}$.

(g) Show that A^{-1} is a cut.

(h) Show that the set of all cuts satisfies the multiplication axioms 5) – 10) and the order axioms 11)–16).

(i) Show that the operations $+$, $-$, $\leq$ and reciprocal on the rational cuts agree with the usual notions of addition etc on the rational numbers.

This shows that the set of cuts is an ordered field, with the rational numbers as a *subfield*. We call each cut a *real number*. This coincides with your intuitive notion of real numbers; for example, we identify $\sqrt{2}$ with $\{q \in \mathbb{Q} : q < \sqrt{2}\}$.

Finally we prove Dedekind's theorem.

(j) Let α and β be disjoint nonempty sets of real numbers (ie sets of cuts) such that every real number (cut) is either in α or in β, and such that if $A \in \alpha$ and $B \in \beta$ then $A < B$. Then there is a unique real number C for which $A \leq C$ for all $A \in \alpha$, and $C \leq B$ for all $B \in \beta$.