# Final Exam, Math 171

Spring, 2002

1. Let $A$ be a subset of a metric space $(M, d)$. We say that $x \in M$ is a limit point of $A$ if for every $\epsilon > 0$ the ball $D(x, \epsilon)$ contains a point of $A$.

   (a) Show that every accumulation point is a limit point. Is the converse true? (Prove or give a counter-example).

   (b) If $x$ is a limit point of $A$, show that there is a sequence $x_n \in A$ with $x_n \to x$.

   (c) Show that a set $A \subseteq M$ is closed if and only if it contains all of its limit points.

2. (a) Give the definition for a set $A$ in a metric space $M$ to be totally bounded.

   (b) What is the relationship between a set being bounded and being totally bounded? (ie does one imply the other?) Prove your assertion, and give a counterexample if the converse fails.

3. (a) Show that a function $f : (M, d) \to (N, \rho)$ is continuous if and only if $f^{-1}(B) \subseteq M$ is open for every open set $B \subseteq N$.

   (b) If $f : M \to N$ is continuous, and $C$ is a open set in $M$, is $f(C)$ a open set in $N$? Prove or give a counter-example. What about if "open" is replaced by "closed"?

4. Let $f_k : (A, d) \to (N, \rho)$ be a sequence of functions.

   (a) Give the definition for the sequence $f_k$ to converge pointwise to a function $f : A \to N$.

   (b) Give the definition for the sequence $f_k$ to converge uniformly to a function $f : A \to N$.

   (c) Give an example of a sequence of functions $f_k$ which converge pointwise to some function $f$, but do not converge uniformly.

   (d) Let $f_k : M \to V$ and $g_k : M \to V$ be two sequences of functions from a metric space $(M, d)$ to a normed space $V$. If $f_k$ converges uniformly to $f$ and $g_k$ converges uniformly to $g$, show that $f_k + g_k$ converges uniformly to $f + g$.

5. Let $C(I)$ be the metric space of continuous functions from $[0, 1]$ to $\mathbb{R}$, with the sup metric. Let $A = \{f \in C(I) : f(x) > 0 \text{ for all } x \in [0, 1]\}$. Show that $A$ is open in $C(I)$.

6. (a) If $f : (M,d) \to (M,d)$ is continuous, show that the *displacement function* $g : (M,d) \to \mathbb{R}$ given by $g(x) = d(x, f(x))$ is continuous. Here $\mathbb{R}$ has the usual metric.

   (b) State the definition for a function $h : (M,d) \to (N,\rho)$ to be uniformly continuous.

   (c) If $f : (M,d) \to (M,d)$ is uniformly continuous, is the displacement function $g : (M,d) \to \mathbb{R}$ uniformly continuous?

7. We say that a function $f : (M,d) \to (M,d)$ is a *contraction* if there exists $0 \leq k < 1$ for which $d(f(x), f(y)) \leq k \cdot d(x,y)$ for all $x, y \in M$.

   (a) State the Contraction Mapping Principle.

   (b) Show that you could not replace the condition that $d(f(x), f(y)) \leq k \cdot d(x,y)$ with the one that $d(f(x), f(y)) < d(x,y)$ in the Contraction Mapping Principle. Hint: Consider the function $f(x) = x + \frac{1}{x}$ on $M = [1, \infty)$.

   (c) Show that we can relax the condition if $M$ is compact; ie Show that if $M$ is compact and $d(f(x), f(y)) < d(x,y)$ for all $x, y \in M$ then $f$ has a fixed point. Hint: Question 6.