MATH 111 HOMEWORK 2

DUE MONDAY, JANUARY 26

(1) (CLO 1.2 #6) Prove that every single point $(a_1, \ldots, a_n) \in k^n$ is an affine variety. Use this to prove that every finite subset of k^n is an affine variety.
(2) (CLO 1.2 #8) Show that $X = \{(x,x) : x \in \mathbb{R}, x \neq 1\} \subset \mathbb{R}^2$ is not an affine variety. (The question in the book has further hints).
(3) (CLO 1.4 #8) An ideal I is *radical* if $f \in I$ if and only if $f^m \in I$ for some positive integer m. Recall $I(V) = \{f \in S : f(x) = 0 \text{ for all } x \in V\}$.
   (a) Prove that $I(V)$ is always a radical ideal.
   (b) Prove that $\langle x^2, y^2 \rangle$ is not $I(V)$ for any affine variety $V \subset k^2$.
(4) (CLO 1.5 #4) If h is the GCD of $f, g \in k[x]$, then prove that there are polynomials $A, B \in k[x]$ such that $Af + Bg = h$.
(5) (CLO 1.5 #11) The *consistency problem* asks, given $f_1, \ldots, f_s \in S$, whether $V(f_1, \ldots, f_s) = \emptyset$.
   (a) Let $f \in \mathbb{C}[x]$ be a nonzero polynomial. Show that $V(f) = \emptyset$ if and only if f is a nonzero constant.
   (b) If $f_1, \ldots, f_s \in \mathbb{C}[x]$, prove that $V(f_1, \ldots, f_s) = \emptyset$ if and only if $GCD(f_1, \ldots, f_s) = 1$.
   (c) Describe (in words) an algorithm to determine whether $V(f_1, \ldots, f_s) = \emptyset$ in this case.
(6) (CLO 1.5 #12-15) Let $f \in \mathbb{C}[x]$.
   (a) Show that f factors completely. That is, we can write $f = c(x - a_1)^{r_1} \ldots (x - a_l)^{r_l}$.
   (b) Show that $V(f) = \{a_1, \ldots, a_l\}$.
   (c) Let $f_{red} = (x - a_1) \ldots (x - a_l)$. Show that $I(V(f)) = \langle f_{red} \rangle$.
   (d) The *formal derivative* of a polynomial $f = a_0 x^n + \ldots a_n \in \mathbb{C}[x]$ is $f' = n a_0 x^{n-1} + \ldots a_{n-1}$. Prove that $GCD(f, f') = (x - a_1)^{r_1 - 1} \ldots (x - a_l)^{r_l - l}$. (The exercise in the book provides more hints)
   (e) Show that $f_{red} = f / GCD(f, f')$. This means that we can compute f_{red} *without* factoring f - purely symbolically!.
   (f) What is $I(V(x^{11} - x^{10} + 2x^8 - 4x^7 + 3x^5 - 3x^4 + x^3 + 3x^2 - x - 1))$? (You may want to use a computer algebra package).