MATH 108, FALL 2002

HOMEWORK 1 SOLUTIONS

These are sample solutions; most of the problems can be solved correctly in more than one way.

(1) Let G be a connected graph on n vertices.

($\Rightarrow$) Assume that G is a tree. We will show by induction on n that G has $n-1$ edges. For $n = 1$, the only tree possible is a single vertex with no edges. Now let $n > 1$, and suppose that every tree on k vertices has $k - 1$ edges for all $k < n$.

Notice that a finite tree with more than one vertex must contain a vertex of degree 1 (often called a *leaf*). To show this, suppose (for contradiction) that the tree does not contain a leaf. Then every vertex has degree at least 2 (a tree is connected, so none of the vertices can have degree 0). But then we can obtain a closed path in G: start by walking across any edge, and then always leave a vertex via an edge other than the edge used to go to the vertex; since there are only finitely many vertices, eventually the same vertex is reached twice. Such a walk yields a closed path in the tree, contradicting it being a tree.

Now pick a leaf l in G, and let G' be the graph obtained by deleting l and its edge from G. Note that G' is a tree, since the deleted edge is not needed to go from $x \neq l$ to $y \neq l$ (if we visited l in G, we would have to turn back and return to the previous vertex anyway since l is a leaf). By the inductive hypothesis, G' has $n - 2$ edges. Hence, G has $n - 1$ edges.

($\Leftarrow$) Now assume that G is not a tree, and show that G does not have $n - 1$ vertices. Since G is connected but not a tree, G must have a simple closed path C. Delete any edge e_1 of C from the graph. The resulting graph G' is still connected since in any walk that relied on e_1, we can detour by going around the rest of C instead of using e_1. If G' is a tree, stop. Otherwise, find a simple closed path in G' and continue as above. This process must terminate in a tree T since there are only finitely many edges in G. The resulting tree T has $n - 1$ edges by the previous part, so G has more than $n - 1$ edges since at least one edge was deleted in the process.

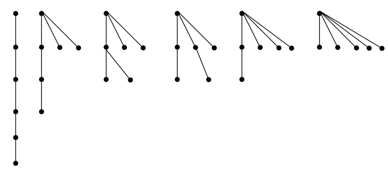
(2) Suppose we can place a_1, a_2, a_3, b_1, b_2 and b_3 in the plane in such a way that the drawing of $K_{3,3}$ is planar. The quadrilateral formed by the edges (a_1, b_1), (a_2, b_1), (a_2, b_2) and (a_1, b_2) divides the plane into two pieces. Since there is an edge between a_3 and b_3, these two points must lie either inside or outside the quadrilateral. Suppose first that they lie inside the quadrilateral. Now the edges (a_3, b_1) and (a_3, b_2) divide the quadrilateral into two sections, one containing a_1, and one containing a_2. We now consider the placement of b_3. If it is in the section containing a_1, then the edge (a_2, b_3) must cross some other edge. If it is in the section containing a_2, then the edge (a_1, b_3) must cross some other edge. So we conclude that a_3 and b_3 lie on the outside of this quadrilateral. But now an almost identical argument shows that this is also imposible, so we conclude that $K_{3,3}$ is not planar.

(3) Let G be a simple graph with n vertices. We use the *pigeonhole principle*: if k objects are placed in m boxes, with $k > m$, then at least one box will contain more than one object. The "objects" are the n vertices and the "boxes" are the possible degrees, which are 0, 1, . . . , n-1. We put a vertex in box d iff the vertex has degree d. There are n vertices and n possible degrees, so we

cannot yet use the pigeonhole principle. However, note that if some vertex has degree 0, then no vertex can have degree n-1. So the number of degrees that occur in G is strictly less than n. Thus by the pigeonhole principle, there are two vertices sharing the same degree.

(4) The i, j entry of A^2 is $\sum_{k=1}^{n} a_{ik}a_{kj}$. The term $a_{ik}a_{kj}$ counts the number of walks of length 2 from i to j through k, since such a walk is obtained by choosing an edge from i to k and then choosing an edge from k to j. Therefore, $\sum_{k=1}^{n} a_{ik}a_{kj}$ counts the number of walks of length 2 from i to j. So the i, j entry of A^2 gives the number of walks of length 2 in the graph from i to j.

(5) The 6 nonisomorphic trees on 6 vertices are drawn below:



Only the second and fourth trees have the same sets of degrees, so only the second and fourth trees could be isomorphic. To see that they are nonisomorphic, note that the 2 vertices of degree 2 are adjacent in the second tree but not in the fourth tree. So these 6 trees are nonisomorphic.

As in Example 2.2, the number of spanning trees in K_6 isomorphic to a specific tree on 6 vertices is 6! divided by the size of the automorphism group of the tree. The first tree has 2 automorphisms (one where the 2 leaves are swapped, and the identity). The second tree also has 2 automorphisms (the degree 3 vertex and the degree 2 vertex adjacent to it must be fixed, but the 2 leaves adjacent to the degree 3 vertex can be swapped). The third tree has $2^3 = 8$ automorphisms (an automorphism of this tree is determined by choosing whether or not to swap the 2 degree 3 vertices, whether or not to swap the 2 children of one of the degree 3 vertices, and whether or not to swap the 2 children of the other degree 3 vertex). Similarly, the fourth tree has 2 automorphisms, the fifth tree has $3! = 6$ automorphisms, and the sixth tree has $5! = 120$ automorphisms.

Thus, the numbers of distinct spanning trees in K_6 isomorphic to these trees are 360, 360, 90, 360, 120, and 6, respectively. We check that this is correct by noting that $360+360+90+360+120+6 = 1296 = 6^4$.

(6) Draw the *dual graph*, whose vertices are the regions of the squiggle, and for which there is an edge if two regions are adjacent. It suffices to show that this graph is bipartite. The dual graph can be drawn as a planar graph, with each region of the new graph corresponding to one place where the squiggle crosses itself. Since the squiggle leaves each crossing it enters, each region of the new graph is surrounded by a polygon with an even number of sides. We now show that this means that the graph is bipartite. We will use the following lemma.

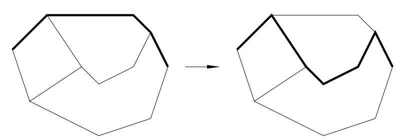
Lemma 1. *A graph is bipartite if and only if it contains no closed path of odd length.*

Proof. Suppose first that the graph is bipartite, so the vertices can be divided into two groups X and Y, and every edge joins a vertex in X to a vertex in Y. Then any path of odd length joins a vertex in X to a vertex in Y, while a path of even length joins a vertex in X to another (possibly identical) vertex in Y or a vertex in Y to a vertex in Y. This means that a closed path must have even length.

Now suppose that G is a graph which is not bipartite. Pick a starting vertex v_0 of the graph, and colour that red. Colour its neighbours blue, and continue, at each stage colouring a yet-uncoloured vertex which is adjacent to a coloured one the opposite colour. Stop if which colour is opposite is ever unclear. If we never get stuck we would have shown that the graph is bipartite, so we will get stuck at some vertex v_1. Then there is a path of alternating colours from v_0 to v_1 which would suggest colouring v_1 blue (so the path has even length). There is also a path of alternating colours from v_0 to v_1 which would suggest colouring v_1 red (so the path has odd length). Cobbling these two paths together gives a closed path of odd length.

□

Suppose now that the dual graph contains a closed path of odd length. We may assume that this path is simple, as if it revisits a vertex either the length of the path up to that point or the length of the path after that point must be odd. Since the graph is planar, this path encloses a number k of regions of the graph. Choose the simple closed path so that k is as small as possible. Since each region is surrounded by an even polygon we know that $k \geq 2$. Now take an edge of the path and notice that we can "push it across" the region it is adjacent to to get a new closed path also of odd length which encloses $k-1$ regions.



This contradicts the fact that k was as small as possible, so we conclude that the path did not exist, so the graph was bipartite and so our squiggle could be coloured with only two colours.

(7) Represent the party-goers as vertices in a graph, and draw a red edge between x and y if x and y know each other, and a blue edge between them otherwise. We need to show that this graph contains a monochromatic triangle (one with edges all red or all blue). Fix a vertex x. There are 5 edges emanating from x, of which either at least 3 are red or at least 3 are blue. Suppose without loss of generality that at least 3 of the edges are red, say $\{x,y\},\{x,z\},\{x,w\}$. If any two of y,z,w have a red edge between them, then x together with those two form a red triangle. Otherwise, y,z,w form a blue triangle. So in all cases, there is a monochromatic triangle, which corresponds to 3 people all of whom know each other or none of whom know each other.

If there are 5 people instead of 6, the claim is false. For example, labelling the vertices of the graph as $1,\ldots,5$, the red edges could be $\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,1\}$ (with all other edges blue), in which case there is no monochromatic triangle.

Now assume we need 4 people such that all know each other or none know each other, i.e., we need a monochromatic copy of K_4. We now show that 20 vertices suffice.

As above, fix a vertex x. Then x has at least 10 red edges or at least 10 blue edges. Assume without loss of generality that x has 10 red edges, say $\{x, y_1\}, \dots, \{x, y_{10}\}$. If there is a red triangle yzw among $y_1, \dots, y_{10}$, then $xyzw$ is a red K_4. So assume there is no red triangle among the y_i.

Now consider the edges between y_1 and all the other y_i. Of these 9 edges, either at least 6 are blue or at least 4 are red. First assume at least 6 are blue, say $\{y_1, z_1\}, \dots, \{y_1, z_6\}$. Then there is a monochromatic triangle among $z_1, \dots, z_6$, which must be blue. This triangle together with y_1 is then a blue K_4. Now assume instead that at least 4 of the edges between y_1 and the other y_i are red, say $\{y_1, w_1\}, \dots, \{y_1, w_4\}$. None of the edges $\{w_i, w_j\}$ can be red since that would give a red triangle. So all of the edges $\{w_i, w_j\}$ are blue, which gives a blue K_4. Thus, in all cases there is a monochromatic K_4.

Remarks: With more work, it can be shown that 18 people suffice for the 4 case, and 18 cannot be improved upon since it is possible to find an example with 17 people where there is no set of 4 people who all know each other or none of whom know each other. Amazingly, the smallest number of people needed for the case of 5 (and higher) is unknown, though progress has been made in finding bounds.

(8) (a) Suppose that $|\{j : \phi(j) \leq i\}| < i$ for some i. Note that if $\phi(j) > i$, then car j parks in a space $> i$. So if car j parks in a space $\leq i$, then $\phi(j) \leq i$, which shows that the set of cars which park in spaces $\leq i$ is a subset of $\{j : \phi(j) \leq i\}$. Therefore, fewer than i cars park in spaces $1, \dots, i$, which implies that not everyone can park.

Conversely, assume that not all cars can park. Then some space i is left empty. Since there is an empty space at i, any car j with $\phi(j) \leq i$ parks at a space $< i$ (since such a car would prefer i to any space $> i$). So the number of cars j with $\phi(j) \leq i$ is less than or equal to the number of cars parked in spaces $1, \dots, i-1$, showing $|\{j : \phi(j) \leq i\}| < i$.

Thus, everyone can park iff $|\{j : \phi(j) \leq i\}| \geq i$ for all i.

(b) As in the hint, suppose that instead of a one-way street the cars park in a circle with a bonus space 0, and the same parking rules (the cars still start at space 1, but can wrap around the circle if necessary). We allow $\phi(j) = 0$. Note that in this scenario, everyone can park for *any* function ϕ, since a car whose preferred space is taken can just continue around the circle until a free space is found.

Moreover, the function ϕ is a parking function (in the original sense) iff the bonus space 0 is not parked in. Let S be the set of all functions ϕ from the set of cars into the set of parking spaces $\{1, \dots, n, 0\}$. Partition S into the sets $S_0, \dots, S_{n-1}$, where S_i is the set of ϕ such that space i does not get parked in. These sets S_i all have the same size since shifting everyone's preferred space by k spaces along the circle also shifts the space that gets left empty by k spaces. So $|S_0| = \frac{1}{n+1}|S| = (n+1)^{n-1}$. Thus, there are $(n+1)^{n-1}$ parking functions in the original sense.

Remarks: The number $n + 1^{n-1}$ should look familiar, as the number of labeled trees on $n+1$ vertices. This suggests searching for a natural bijection between parking functions and trees. Such bijections have been found, although it may require a lot of work to prove that they are bijections.