# MATH 108, FALL 2002

## HOMEWORK 7 SOLUTIONS

(1) Represent the grid coloring as an $n$ by $n$ matrix $A = (a_{ij})$ by putting $a_{ij} = 1$ if the square in row $i$, column $j$ is blue, and $a_{ij} = 0$ if that square is red. Then the entries of $A$ are nonnegative integers with all row and column sums equal to $n/2$, so we can apply Birkhoff's Theorem to write $A = P_1 + \cdots + P_{n/2}$ as a sum of $n/2$ permutation matrices.

Let $v(i,j) = (i-1)n + j$ be the value written in the square in row $i$, column $j$. Let $b$ be the sum of the values on the blue squares and $r$ be the sum of the values on the red squares. Then

$$r + b = \sum_{i,j} v(i,j) = \sum_{i,j} ((i-1)n + j) = \frac{n^4 + n^2}{2}.$$

Note that if $a_{ij} = 1$, then exactly one of the permutation matrices $P_k$ has a 1 in row $i$, column $j$. So $b = \sum_{k=1}^{n/2} c_k$, where $c_k$ is the sum of the values $v(i,j)$ in the positions where $P_k$ has 1's. But if $P = (p_{ij})$ is a permutation matrix, corresponding to a permutation $\sigma$ (so $p_{ij} = 1$ iff $\sigma(j) = i$), then the sum of the $v(i,j)$ where $P$ has 1's is

$$\sum_{j=1}^{n} v(\sigma(j), j) = \sum_{j=1}^{n} ((\sigma(j) - 1)n + j) = \frac{n(n+1)}{2} + \frac{n^2(n-1)}{2} = \frac{n^3 + n}{2}$$

since $\sigma$ is a permutation. Thus,

$$b = \frac{n}{2}\frac{n^3 + n}{2} = \frac{n^4 + n^2}{4} = \frac{r + b}{2},$$

showing $b = r$.

(2) Let $V_1, \ldots V_k \in P$ and check that they have a greatest lower bound and a least upper bound. The intersection $V_1 \cap \ldots V_k$ is a subspace contained in all the $V_i$ and containing any subspace which contains all the $V_i$, so the intersection is a greatest lower bound. Similarly, the sum $V_1 + \cdots + V_k = \{v_1 + \cdots + v_k : v_i \in V_i\}$ is a least upper bound for the $V_i$. So $P$ is a lattice.

Let $V_0 \subset V_1 \cdots \subset V_k$ be a maximal chain (with the $V_i$ distinct). Then the dimension of $V_{i+1}$ is 1 more than the dimension of $V_i$ since otherwise we can make the chain longer (by choosing a basis for $V_i$, extending it to a basis for $V_{i+1}$, and inserting a subspace which uses just one of the new basis elements). Also, $V_0$ is 0-dimensional and $V_k$ is $d$-dimensional (else we can get a longer chain using $\{0\}$ or the whole space). Thus, $k = d$, showing that the chain has $d + 1$ elements and length $d$ (depending on the convention used for length).

As shown on p. 326 of the text, the number of $k$-subspaces is the Gaussian coefficient

$$\binom{d}{k}_p = \frac{(p^d - 1)(p^{d-1} - 1)\ldots(p^{d-k+1} - 1)}{(p^k - 1)(p^{k-1} - 1)\ldots(p - 1)}.$$

(3) Let $V$ be the given $r$-subspace of an $n$-dimensional space $V_0$, and define $f(U)$ to be the number of $k$-subspaces $S$ such that $S \cap V = U$ (in particular, $f(U) = 0$ unless $U \subseteq V$). Define

$$h(W) = \sum_{U \supseteq W} f(U).$$

Then $h(W)$ counts the number of $k$-subspaces $S$ with $S \cap V \supseteq W$. So $h(W) = 0$ if $W$ is not contained in $V$ or if $\dim(W) > k$. Assume $W \subseteq V$ and $\dim(W) = j \leq k$. Then $h(W) = \binom{n-j}{k-j}_q$ (this is a Gaussian coefficient as in the previous problem), by directly counting or since the lattice of all subspaces containing $W$ is isomorphic to the quotient space $V_0/W$. Applying Möbius inversion, we have

$$f(U) = \sum_{W \supseteq U} \mu(U, W) h(W).$$

Taking $U = \{0\}$ and using Theorem 25.1 (iii),

$$f(\{0\}) = \sum_{W} (-1)^{\dim(W)} q^{\binom{\dim(W)}{2}} h(W) = \sum_{j=0}^{\min(r,k)} (-1)^j q^{\binom{j}{2}} \binom{r}{j}_q \binom{n-j}{k-j}_q$$

is the number of $k$-subspaces which intersect $V$ trivially.