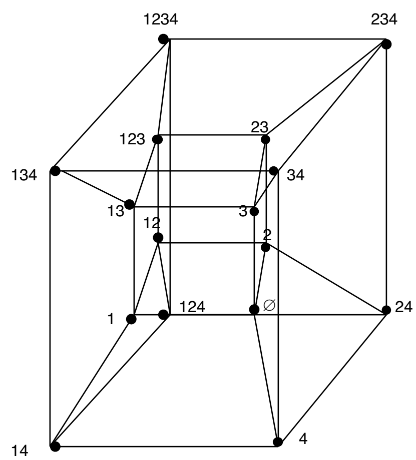
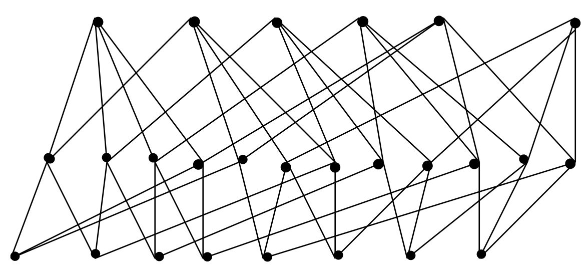
MATH 108, FALL 2002

HOMEWORK 6 SOLUTIONS

(1) There are $2^4 = 16$ subsets of $\{1,2,3,4\}$. To see the correspondence with the hypercube, identify a subset S with a 4-tuple (a_1, a_2, a_3, a_4) where a_i is 1 if $i \in S$ and is 0 otherwise. In the labels below we just write 1234 in place of $\{1,2,3,4\}$, etc.



(2) There are 6 maximal elements (the faces in the usual sense), each of which lies above the 4 edges it contains, each of which lies above 2 minimal elements (vertices). Each of the 8 minimal elements is contained in 3 edges, each of which is contained in 2 maximal elements.



(3) There are $\binom{n}{3}$ ways to select the 3 non-leaves. Now count the number of trees for a particular choice of the 3 non-leaves, which (without loss of generality) we assume are 1,2,3. Recalling that a vertex v of a tree appears $\deg(v) - 1$ times in the tree's Prüfer code, we must count the number of sequences $(a_1, \ldots, a_{n-2})$ with $a_i \in \{1,2,3\}$ and where $1, 2$, and 3 all appear at least once. Let $S = \{1,2,3\}^{n-2}$ and A_i be the number of sequences in S where i does not appear. Then by inclusion-exclusion,

$$\begin{aligned} |S \setminus (A_1 \cup A_2 \cup A_3)| &= |S| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3| \\ &= 3^{n-2} - 3 \cdot 2^{n-2} + 3 \cdot 1 - 0 \\ &= 3(3^{n-3} - 2^{n-2} + 1) \end{aligned}$$

Thus, there are $3\binom{n}{3}(3^{n-3}-2^{n-2}+1)$ labelled trees on the vertices $\{1,\ldots,n\}$ with exactly $n-3$ leaves.

(4) Define a partial order $\leq$ on the set of all pairs (i,a_i) $(i\in\{1,2,\ldots,n^2+1\})$ by saying $(i,a_i)\leq(j,a_j)$ iff $i\leq j$ and $a_i\leq a_j$. (It is easy to check that this relation is reflexive, transitive, and antisymmetric.) Note that a chain of size m in this poset corresponds to an increasing subsequence of length m, since if $\{(i_1,a_{i_1}),\ldots,(i_m,a_{i_m})\}$ is a chain of size m, labeled so that $(i_1,a_{i_1})\leq(i_2,a_{i_2})\leq\cdots\leq(i_m,a_{i_m})$, then $a_{i_1}<a_{i_2}<\ldots a_{i_m}$ with $i_1<i_2<\cdots<i_m$ (the inequalities are strict since $a_1,\ldots,a_{n^2+1}$ are distinct). Similarly, an antichain of length m corresponds to a decreasing subsequence of length m.

If there is an antichain of size $n+1$, then we are done since this gives a monotone decreasing subsequence of length $n+1$. So assume that all antichains have size at most n. By Dilworth's Theorem, we can partition the poset into n chains $C_1,\ldots,C_n$ (some of these chains may be empty). Then by the Pigeonhole Principle, some C_i has size at least $n+1$, which gives a monotone increasing subsequence of length $n+1$.

(5) Let $S=\{1,2,\ldots,999\}$ and let A_i be the set of elements of S that are divisible by i. An integer has no factor between 1 and 10 iff it has no prime factor between 1 and 10, so we need to count $S\setminus(A_2\cup A_3\cup A_5\cup A_7)$. Note that if $i_1,\ldots,i_k$ are *distinct* primes then $A_{i_1}\cap\cdots\cap A_{i_k}=A_{i_1i_2\ldots i_k}$. Also, the size of A_i is $\lfloor 999/i\rfloor$. So by inclusion-exclusion the desired number is

$$|S|-(|A_2|+|A_3|+|A_5|+|A_7|)+(|A_6|+|A_{10}|+|A_{14}|+|A_{15}|+|A_{21}|+|A_{35}|)-(|A_{30}|+|A_{42}|+|A_{70}|+|A_{105}|)+|A_{210}|=$$

$$999-(499+333+199+142)+(166+99+71+66+47+28)-(33+23+14+9)+4=228$$

(6) Fix $n\geq 1$ and p prime. Let S be the set of all monic polynomials of degree n in $\mathbb{F}_p[x]$ and for $i\in\mathbb{F}_p$, let A_i be the set of all polynomials $f(x)$ in S with $f(i)=0$. We want to count $S\setminus\bigcup_{i=0}^{p-1}A_i$. Now use the fact that $f(i)=0$ iff $x-i$ divides $f(x)$. A polynomial $f(x)$ in S is divisible by $x-i$ iff $f(x)=(x-i)g(x)$ where $g(x)$ is monic of degree $n-1$. And $f(x)$ is divisible by $(x-i)(x-j)$ (where $i\neq j$ iff $f(x)=(x-i)(x-j)g(x)$ with $g(x)$ monic of degree $n-2$, etc. (using the fact that $\mathbb{F}_p[x]$ has unique factorization; also, $g(x)$ is uniquely determined since if $(x-i)(x-j)g_1(x)=(x-i)(x-j)g_2(x)$ then $(x-i)(x-j)(g_1(x)-g_2(x))=0$, which implies $g_1(x)-g_2(x)=0$ since all nonzero elements of $\mathbb{F}_p$ have inverses). By inclusion-exclusion, the desired number is

$$p^n-\sum_{j=1}^{\min(p,n)}(-1)^{j+1}\binom{p}{j}p^{n-j}=\sum_{j=2}^{\min(p,n)}(-1)^j\binom{p}{j}p^{n-j}.$$