MATH 108, FALL 2002

HOMEWORK 5 SOLUTIONS

(1) Let X be a set of 4 points in $\mathbb{F}_3^2$ not containing any lines. First consider the case where X contains $(0,0)$, say with $X = \{(0,x,y,z\}$. The points x,y,z are linearly dependent since $\mathbb{F}_3^2$ is a 2-dimensional vector space. So there is a nontrivial linear combination $\alpha x + \beta y + \gamma z = (0,0)$ with $\alpha, \beta, \gamma \in \{-1,0,1\}$ (working mod 3). None of the coefficients can be 0, else X would contain a line (e.g., if $\alpha = 0$ then X contains the line $\{0, y, -y\}$). Similarly, the coefficients cannot be all 1's or all -1's. So rearranging $\alpha x + \beta y + \gamma z = 0$, we can write one of x, y, z is the sum of the other two. Without loss of generality, assume $z = x + y$.

Then we can take $b = 0$ and $A = (x\ y)$ (treating points as column vectors) since $0 \mapsto 0, (1,0) \mapsto x, (0,1) \mapsto y$, and $(1,1) \mapsto x + y = z$. The matrix A is invertible since x and y are linearly independent (since X does not contain a line).

If X does not contain $(0,0)$, then we can translate the elements of X. Specifically, let $X = \{p_1, p_2, p_3, p_4\}$. Let $b = p_1$ and let A be the matrix obtained from applying the previous case to $X' = \{0, p_2 - p_1, p_3 - p_1, p_4 - p_1\}$ (which also does not contain a line). Then the affine transformation $v \mapsto Av + b$ maps $\{(0,0),(1,0),(0,1),(1,1)\}$ onto X.

(2) Let $(a_1, \ldots, a_{10})$ be an ISBN codeword. Here $a_i \in \{0,1,\ldots,9\}$ for $i < 10$, and $a_{10} \in \{0,1,\ldots,10\}$ is chosen so that

$$10a_1 + 9a_2 + \ldots + 2a_9 + a_{10} = \sum_{i=1}^{10} (11-i)a_i \equiv 0 (\text{mod } 11).$$

Equivalently, the check condition is $\sum_{i=1}^{10} ia_i \equiv 0(\text{mod } 11)$.

Suppose that there is exactly one error, at the jth position, which gives the sequence $(b_1, \ldots, b_{10})$ where $b_j \neq a_j$ and $b_i = a_i$ for $i \neq j$. Then

$$\sum_{i=1}^{10} ib_i \equiv \sum_{i=1}^{10} ib_i - \sum_{i=1}^{10} ia_i = \sum_{i=1}^{10} i(b_i - a_i) = j(b_j - a_j),$$

which is nonzero (mod 11) since j and $b_j - a_j$ are nonzero (mod 11). (We are using the fact that 11 is prime and that if $ab \equiv 0(\text{mod } p)$ with p prime, then $a \equiv 0(\text{mod } p)$ or $b \equiv 0(\text{mod } p)$.) Thus, the error is detected.

Now suppose instead that two adjacent numbers are switched, say a_j and a_{j+1}. This does not affect the codeword if $a_j = a_{j+1}$, so assume $a_j \neq a_{j+1}$ and let $(b_1, \ldots, b_{10})$ be the resulting word. Then

$$\sum_{i=1}^{10} ib_i \equiv \sum_{i=1}^{10} i(b_i - a_i) = j(b_j - a_j) + (j+1)(b_{j+1} - a_{j+1}) = j(a_{j+1} - a_j) + (j+1)(a_j - a_{j+1}) = a_j - a_{j+1},$$

which is nonzero (mod 11). So the switch is detected.

(3) We can use the *repetition code* $C = \{(0,0,0,0),(1,1,1,1),(2,2,2,2)\}$. This has weight 4 (in fact both nonzero codewords here have weight 4), with 3 codewords.

(4) Let C be the code determined by the 4 by 13 parity check matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

So C is the set of all row vectors $x \in \mathbb{F}_3^{13}$ such that $Hx^T = 0$. We chose H so that any 2 columns are linearly independent over $\mathbb{F}_3$ (it is easy to see that the H above has this property since the columns are nonzero, distinct, and none is the negative of another). On the other hand, it is possible to find 3 linearly dependent columns in H, such as the 9th, 10th, and 11th columns. This shows that C has weight 3, since for any row vector $x \in \mathbb{F}_3^{13}$, Hx^T is a linear combination of the columns of H. There are 3^9 codewords because the dimension of the nullspace of H is $13 - \text{rank}(H) = 13 - 4 = 9$.