MATH 108, FALL 2002

HOMEWORK 4 SOLUTIONS

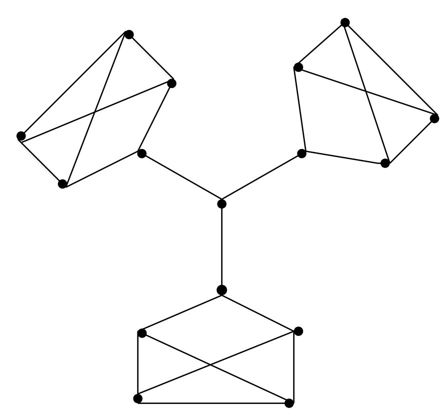
(1) (i) Let G be a finite bipartite graph which is regular of degree d, with vertex set $X \cup Y$ such that every edge has one endpoint in X and the other in Y. Note that

$$d|X| = \sum_{x \in X} \deg(x) = (\text{number of edges in the graph}) = \sum_{y \in Y} \deg(y) = d|Y|,$$

so X and Y have the same size.

Let $A \subseteq X$, and check the condition for Hall's Marriage Theorem. Every edge with an endpoint in A also has an endpoint in $\Gamma(A)$. So the number of edges with an endpoint in A is less than or equal to the number of edges with an endpoint in $\Gamma(A)$. This gives $d|A| \leq d|\Gamma(A)|$, so $|\Gamma(A)| \geq |A|$. By Hall's Marriage Theorem, there is a complete matching M of X into Y. This M is a perfect matching of G since X and Y have the same size, so every vertex in X is matched with a unique vertex in Y and vice versa.

(ii) Note that if a finite graph has a perfect matching, then the number of vertices must be even. For a trivalent simple graph $G = (V, E)$ with n vertices, $3n = 2|E|$ so n is even. This rules out giving an example where no perfect matching exists due to an odd number of vertices. However, we can find a trivalent simple graph where a perfect matching would give rise to a perfect matching in a subgraph with an odd number of vertices:



The graph drawn above does not have a perfect matching. To see this, suppose there is a perfect matching. Then exactly one of the 3 edges coming out from the center vertex is used in the matching. If we delete the other 2 edges, the graph is broken into 3 connected components. The perfect matching induces perfect matchings on the components, but this is impossible since 2 of the components have odd numbers of vertices.

(iii) Assume $s \geq t$ (this is equivalent to $|X| \leq |Y|$ since $s|X| = t|Y|$). By the same method as in (i), we have $s|A| \leq t|\Gamma(A)|$ for any $A \subseteq X$. Then $|\Gamma(A)| \geq \frac{s}{t}|A| \geq |A|$, so Hall's Marriage Theorem yields a complete matching of X into Y.

(2) It suffices to show that if we have five points $p_1, \ldots, p_5$ in $\mathbb{P}_2^2$, then at least three are collinear. For each pair of points p_i, p_j (with $i < j$), there is a unique line $L(i,j)$ through p_i and p_j. These lines are not all distinct since there are $\binom{5}{2} = 10$ pairs but only 7 lines in $\mathbb{P}_3^2$. So $L(i,j) = L(k,l)$ for some i,j,k,l with $\{i,j\} \neq \{k,l\}$. But then the points p_i, p_j, p_k, p_l are collinear (at least 3 of these points are distinct since $\{i,j\} \neq \{k,l\}$). This means that we have three collinear points, which contradicts $p_1, \ldots, p_5$ being a cap.

(3) The 13 points in $\mathbb{P}_3^2$ are the lines through the origin containing the following points:

a) (1,0,0)
b) (0,1,0)
c) (1,1,0)
d) (1,2,0)
e) (0,0,1)
f) (1,0,1)
g) (2,0,1)
h) (0,1,1)
i) (1,1,1)
j) (2,1,1)
k) (0,2,1)
l) (1,2,1)
m) (2,2,1)

The lines are: $\{a,b,c,d\}, \{a,e,h,k\}, \{a,f,i,l\}, \{a,g,j,m\}, \{b,e,f,g\}, \{b,h,i,j\}, \{b,k,l,m\}$, $\{c,e,i,m\}, \{c,f,j,k\}, \{c,g,h,l\}, \{d,e,j,l\}, \{d,f,h,m\}, \{d,g,i,k\}$.

Suppose that $p_1, \ldots, p_5$ are 5 points in $\mathbb{P}_3^2$. Since multiplying by an invertible 3×3 matrix takes lines in $\mathbb{F}_3^3$ to lines, and two-dimensional subspaces to two-dimensional subspaces, and p_1, p_2, p_3 are linearly independent vectors (since they do not lie on the same projective line), we can multiply by $(p_1 p_2 p_3)^{-1}$ to get a new configuration with $p_1' = e_1$, $p_2' = e_2$, $p_3' = e_3$. If $p_1', \ldots, p_5'$ is a cap, the last two points must be two of $\{(1,1,1), (1,2,1), (1,1,2), (1,2,2)\}$. However for each of the six ways to choose two of these points one of p_1', p_2', or p_3' lies on the same line as the two points, so the five points do not form a cap. Thus the original five points do not form a cap.

(4) We can form a cap in $\mathbb{P}_3^3$ by taking a cap in $\mathbb{F}_3^3$ and appending the last coordinate equal to one to get points in $\mathbb{F}_3^4$ which correspond to different points in $\mathbb{P}_3^3$ with no three collinear. For example: $\{(0,0,0,1), (2,0,0,1), (0,2,0,1), (2,2,0,1), (1,1,1,1), (1,0,2,1), (0,1,2,1), (2,1,2,1), (1,2,2,1)\}$.

A cap with ten points is given by $\{(1,1,1,0), (1,1,0,1), (1,0,1,1), (0,1,1,1), (2,1,1,1), (1,2,1,1)$, $(0,0,1,2), (2,0,1,2), (0,2,1,2), (2,2,1,2)\}$.