MATH 108, FALL 2002

HOMEWORK 3 SOLUTIONS

(1) We have $N(4,4;2) \leq N(3,4;2) + N(4,3;2) = 9 + 9 = 18$. To show equality holds, we need an example of a coloring of K_{17} with no monochromatic K_4.

Using the vertex set $\mathbb{Z}_{17}$, color $\{i,j\}$ red iff $i-j$ is $\pm 1, \pm 2, \pm 4$, or ± 8 (mod 17). We just need to check that this has no monochromatic K_4.

Suppose there is a monochromatic K_4. Note that the coloring is translation invariant, in that the color of $\{i,j\}$ is the same as the color of $\{i+x, j+x\}$ for any x. So we can assume 0 is in the vertex set of the K_4. Also, note that $\{3i, 3j\}$ has the opposite color as $\{i,j\}$. So we can assume the K_4 is red. But there is no choice of three numbers from $\pm 1, \pm 2, \pm 4$, or ± 8 so that the differences stay within that list of numbers. So there is no monochromatic K_4.

For the other part, $N(3,5;2) \leq N(2,5;2) + N(3,4;2) = 5 + 9 = 14$. To show equality holds, take $\mathbb{Z}_{13}$ and color $\{i,j\}$ red iff $i-j$ is ± 1 or ± 5 (mod 13).

This coloring has no red K_3 or blue K_5. If there is a red K_3, there is one containing 0, but it is easy to check that there is no such triangle. Assume there is a blue K_5. Again we can assume it contains 0, and write the vertices as $0 < a < b < c < d$, by choosing the representation between 0 and 12 of integers mod 13. Then $a \geq 2, b \geq 4, c \geq 6, d \geq 8$ since no difference can be 1. But $d \neq 8$ since $8 \equiv -5$ (mod 13). Also, $d \neq 12$ since $12 \equiv -1$ (mod 13). Similarly, each of the 3 cases with $9 \leq d \leq 11$ can be ruled out, by considering the possible values for a, b, c.

(2) Color the edges of K_{17} with 3 colors, say red, blue, and green. Pick a vertex v. Of the 16 edges with v as an endpoint, there must be 6 of the same color, say green. Let W be a set of 6 vertices where the edges $\{v, w\}$ are green for $w \in W$. If any two vertices within W are connected by a green edge, they form a green triangle with v. Otherwise, the edges within W are all red or blue, so there is a monochromatic triangle since $R(3,3;2) = 6$.

(3) Let $n = N(\underbrace{3,3,\ldots,3}_{r};2) - 1$. Given a coloring of the integers from 1 to n with r colors, form the complete graph with vertices $1, \ldots, n+1$ and color $\{i,j\}$ with the same color as $|i-j|$. By Ramsey's theorem, there is a monochromatic triangle, say with vertices $\{i,j,k\}$ where $i < j < k$. For this triangle to be monochromatic means that $j-i, k-j$, and $k-i$ have the same color. Taking $x = j-i, y = k-j, z = k-i$, we have x, y, z of the same color with $x + y = z$. Therefore, $N(r)$ exists and satisfies $N(r) \leq N(\underbrace{3,3,\ldots,3}_{r};2) - 1$. We have $N(2) \geq 5$ because of the coloring $\begin{matrix} R & B & B & R \\ 1 & 2 & 3 & 4 \end{matrix}$. To show that $N(2) \leq 5$, suppose there is a 2-coloring of $1,2,3,4,5$ with no such x, y, z. Assume without loss of generality that 1 is colored red. Then 2 is blue since $1+1=2$. So 4 is red since $2+2=4$. Then 3 is blue since $1+3=4$. But then 5 can't be colored without avoiding such an x, y, z since $1+4=5=2+3$.

Thus, $N(2) = 5$ (note that this also equals $N(3,3;2) - 1$).

To show that $N(3) > 13$, we can find a 3-coloring of $\{1, \ldots, 13\}$ with no such x, y, z:

$$\begin{array}{ccccccccccccc} R & B & B & R & G & G & G & G & G & R & B & B & R \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{array}$$

(4) There are 4 forms the desired submatrix can take, depending on whether it has 0's or 1's above the diagonal, and whether it has 0's or 1's below the diagonal. This suggests using 4 colors. Let $A = (a_{ij})$ be an $n \times n$ binary matrix, with $n \geq N(m,m,m,m;2)$. In K_n, color $\{i,j\}$ with the pair (a_{ij}, aji) (for $i < j$). That is, the 4 "colors" are $(0,0), (0,1), (1,0)$, and $(1,1)$. By Ramsey's Theorem, there is a monochromatic K_m, say with vertex set I. The principal submatrix determined by I is of the desired form since $(a_{ij}, a_{ji}) = (a_{kl}, a_{kl})$ for $i,j,k,l \in I$ with $i < j, k < l$.

(5) Let H be a maximal-size set of vertices such that the subgraph induced by H does not contain a red triangle. It suffices to show $k \geq \lfloor \sqrt{2n} \rfloor$, where k is the size of H.

By maximality, for any vertex v outside of H, there is a red triangle v, h, i with $h, i \in H$. Define a function f from the vertices outside of H to the edges in H by assigning an edge $\{h, i\}$ contained in H to each $v \notin H$, such that v, h, i is a red triangle. (Choose any h, i that works for v if there is more than one.) Since no red edge is in more than one red triangle, f is one-to-one (i.e., no edge $\{h, i\}$ is used twice).

Therefore, $n - k \leq \binom{k}{2} = \frac{k(k-1)}{2}$. This rearranges to $k^2 + k \geq 2n$, which gives $\sqrt{k(k+1)} \geq \sqrt{2n}$. We would be done if the $k+1$ were a k, so let us bound $\sqrt{k+1} - \sqrt{k}$:

$$\sqrt{k+1} - \sqrt{k} = \frac{1}{\sqrt{k+1} + \sqrt{k}} < \frac{1}{\sqrt{k}},$$

so

$$\sqrt{2n} \leq \sqrt{k}\sqrt{k+1} < \sqrt{k}(\sqrt{k} + \frac{1}{\sqrt{k}}) = k+1.$$

This implies $k \geq \lfloor \sqrt{2n} \rfloor$ since $\lfloor \sqrt{2n} \rfloor$ is the unique integer in the interval $(\sqrt{2n} - 1, \sqrt{2n}]$.

(6) First consider the case where $n = 2k$ is even, and use induction on k. Note that $\lfloor n^2/4 \rfloor = k^2$. If G has 2 vertices and 1 edge, then G is isomorphic to $K_{1,1}$. Assume that a simple graph on $2k$ vertices with at least k^2 edges and no triangles is isomorphic to $K_{k,k}$. Let G have $2(k+1)$ vertices and at least $(k+1)^2$ edges, with no triangle.

As in the proof of Theorem 4.1, choose a K_{p-1} in G. Since $p = 3$ here, this just means choose any edge $\{a, b\}$. Let G' be the graph obtained by deleting a and b (and all of their edges). Since G has no triangles, no vertex in G' can have edges going to both a and b. So G' has at least $(k+1)^2 - 1 - 2k = k^2$ edges. By the inductive hypothesis, G' is isomorphic to $K_{k,k}$. So G' has exactly k^2 edges, and then every vertex in G' must be connected to a or b by an edge (but not both). Write the vertex set of G' as $X \cup Y$ with X and Y disjoint and no edge from X to X or Y to Y. Then since G has no triangles, either there are no edges from X to a or there are no edges from Y to a. Assume (without loss of generality) there are no edges from X to a. Then there are no edges from Y to b (else we would have a triangle). Adjoining a to X and b to Y, we have G isomorphic to $K_{k+1,k+1}$.

The case with $n = 2k+1$ is essentially the same. Here, $\lfloor n^2/4 \rfloor = k^2 + k$. The case $k = 1$ is trivial. Defining G' as in the even case, G' has at least $(k+1)^2 + (k+1) - 1 - (2k+1) = k^2 + k$ edges, so the induction goes through as in the even case.

(7) Form the graph whose vertices are the cubes and with an edge connecting two cubes iff they are adjacent. This graph is bipartite, i.e., we can color the cubes red and blue with adjacent cubes different colors. The color of the center cube in such a coloring determines all the colors.

Say the center cube is blue. Then there are 14 red cubes and 13 blue cubes. The mouse alternates between reds and blues. If the mouse eats a red cube on the first day, then the mouse can eat blue cubes only on even-numbered days, and so can't eat the blue center cube on the 27th day. On the other hand, if the mouse eats a blue cube on the first day, then there are too few blue cubes for it to be possible to eat the blue center cube on the 27th day, since there are 14 odd-numbered days between 1 and 27 and only 13 blue cubes. Hence, the mouse can't eat the center cube on the 27th day.