MATH 108, FALL 2002

HOMEWORK 2 SOLUTIONS

(1) Recall that vertex v appears $\deg(v) - 1$ times in the Prüfer code of T. So T has the desired degrees iff its Prüfer code is some permutation of (2,2,3,3,5). Thus, there are $\frac{5!}{2!2!} = 30$ such trees.

(2) We will show by induction that every T_k can be extended to a cheapest spanning tree (i.e., that T_k is a subgraph of some cheapest spanning tree). It follows that T_n is a cheapest spanning tree since T_n is a subgraph of a cheapest spanning tree T, and then $T_n = T$ because T_n is already a spanning tree.

For $k = 1$, T_1 can be extended to a cheapest spanning tree since T_1 is just one vertex with no edges, so it is a subgraph of any cheapest spanning tree.

Suppose that T_k can be extended to a cheapest spanning tree C, and show that T_{k+1} can also be extended to a cheapest spanning tree. Let e be the new edge added to T_k to obtain T_{k+1}. If e is already an edge of C then we are done. So assume e is not an edge of C.

Add the edge e into C; this introduces a simple closed path. Since e goes from a vertex in T_k to a vertex not in T_k, this closed path must contain an edge e' from a vertex not in T_k back into T_k. By the description of the algorithm, we have $c(e) \le c(e')$. So the tree C' obtained from C by replacing e' with e is a cheapest spanning tree containing T_{k+1} as a subgraph.

By induction, T_n can be extended to a cheapest spanning tree. Thus, T_n is a cheapest spanning tree.

(3) Let T be a path-graph with n vertices. Without loss of generality, we can assume the edges of T are $\{1, 2\}, \{2, 3\}, \ldots, \{n - 1, n\}$. Label vertex 1 as 1, vertex 2 as n, vertex 3 as 2, vertex 4 as $n-1$, etc. That is, vertex $2j-1$ is labeled as j ($1 \le j \le \lceil n/2 \rceil$) and vertex $2j$ is labeled as $n-j+1$ ($1 \le j \le \lfloor n/2 \rfloor$).

The differences in labels across edges are then $n - 1, n - 2, \ldots, 1$ in absolute value, so the labeling is graceful.

(4) Let n be the number of vertices in G. By Problem 1C, G has $n-1$ edges. There are $n-(m-1)$ monovalent vertices, so (1.1) gives

$$2(n - 1) = n - (m - 1) + \sum_{i=2}^{m} i.$$

Solving for n, we see that

$$n = \frac{m^2 - m}{2} + 2.$$

(5) As suggested in the book's hint, let us first check that given such a graph G, there is a graph G' with the same number of vertices of each degree, and such that the vertex of degree m has edges to the vertices of degrees $2, \ldots, m$.

Let G be such a graph and let v_m be its vertex of degree m. Suppose there is a vertex w with $\deg(w) > 1$ and $\{v_m, w\}$ not an edge (if no such w exists, take $G' = G$. We can rearrange edges as follows:

1. Introduce an edge from v_m to w
2. Delete an edge $\{v_m, l\}$ where l is a leaf.
3. Delete an edge $\{w, x\}$ where $x \neq v_m$.
4. Introduce an edge from x to l.

Note that no vertex's degree is changed by this procedure, and now v_m has an edge to w. Repeating this procedure until v_m has edges to all the other vertices of degree > 1, we reach a graph G' as desired.

Next, we recursively construct such a graph for every $m \geq 2$. For $m = 2$, there is a unique such graph up to isomorphism: a path-graph with 3 vertices. Note that $k = 2 = \lfloor \frac{m+3}{2} \rfloor$ in this case. For $m = 3$, there are two non-isomorphic graphs that work, one with the degree 3 vertex adjacent to the degree 2 vertex, and the other with the degree 3 and degree 2 vertex in separate components. In the first case, $k = 3 = \lfloor \frac{m+3}{2} \rfloor$, and in the second case, $k = 5 > \lfloor \frac{m+3}{2} \rfloor$.

Now assume such a graph G_m has been constructed for m, and construct such a graph G_{m+2} for $m + 2$. Introduce three new vertices v_{m+2}, l_1, and l_2. Draw edges from v_{m+2} to the $m - 1$ vertices of degree > 1. Also draw edges from v_{m+2} to one of the leaves adjacent to v_m and from v_{m+2} to l_1 and l_2. The resulting graph G_{m+2} has the desired degrees. Thus, such graphs exist for all $m \geq 2$.

We will prove the bound $k \geq \lfloor \frac{m+3}{2} \rfloor$ by induction on m. We have already checked the base cases $m = 2$ and $m = 3$. Assume the bound holds for m, and prove it for $m + 2$. Let G be such a graph with maximum degree $m + 2$. As explained above, we can assume that the vertex v_{m+2} of degree $m + 2$ in G is connected to all the other vertices of degree > 1. Form a graph G' by deleting v_{m+2} and the two leaves adjacent to it from G (and deleting all edges incident to v_{m+2}). Then G' is such a graph with maximum degree m, so G' has at least $\lfloor \frac{m+3}{2} \rfloor$ leaves. In going from G to G', two leaves were deleted but the vertex of degree 2 was turned into a leaf. Hence, G has at least $\lfloor \frac{m+3}{2} \rfloor + 1 = \lfloor \frac{(m+2)+3}{2} \rfloor$ leaves, which completes the induction.

(6) Suppose (for contradiction) that $N(p, q; 2) = N(p - 1, q; 2) + N(p, q - 1; 2)$ with both terms on the right even, say $N(p - 1, q; 2) = 2a, N(p, q - 1; 2) = 2b$. Let G be the complete graph on $2a + 2b - 1$ vertices, with each edges colored red or blue. It suffices (by minimality in the definition of $N(p, q; 2)$ to show that G contains a red K_p or a blue K_q.

So assume that G does not contain a red K_p or a blue K_q. Define $r(v)$ to be the red-degree of vertex v (the number of red edges incident with v) and $b(v)$ to be the blue-degree of v.

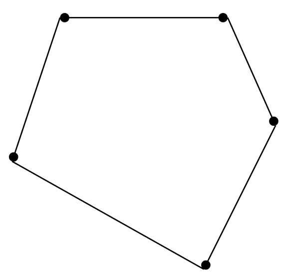
First suppose that some vertex v has $r(v) \geq 2a$. Let H be the subgraph of G induced by the set of vertices w with $\{v, w\}$ a red edge. Then H contains a red K_{p-1} or blue K_q. But a blue K_q contradicts G not having a blue K_q, and a red K_{p-1} in H creates a red K_p in G (by adding v back in), contradicting G not having a red K_p. Therefore, every vertex v satisfies $r(v) \leq 2a - 1$. Similarly, every vertex v satisfies $b(v) \leq 2b - 1$.

On the other hand, $r(v) + b(v) = 2a + 2b - 2$, so we must have $r(v) = 2a - 1, b(v) = 2b - 1$ for all v. But then $\sum_v r(v) = (2a - 1)(2a + 2b - 1)$ is odd, which contradicts $\sum_v r(v)$ being twice the number of red edges in G.

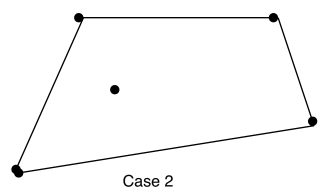
Thus, strict inequality holds in $N(p, q; 2) \leq N(p-1, q; 2) + N(p, q-1; 2)$ if both terms on the right are even.

(7) Let H be the convex hull of the 5 points, and let k be the minimum size of a subset of the 5 points with convex hull containing all 5 points. There are 3 possible cases: $k = 5$ (which makes H a pentagon), $k = 4$ (which makes H a quadrilateral), and $k = 3$ (which makes H a triangle).

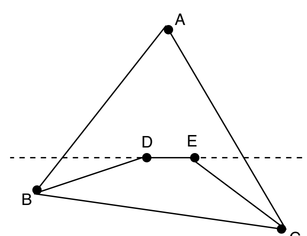
These three cases are illustrated below:



Case 1



Case 2



Case 3

Case 1: $k = 5$. In this case, the 5 points determine a convex pentagon. So any 4 of the points determine a convex quadrilateral.

Case 2: $k = 4$. In this case, the convex hull of 4 of the points contains the fifth point. Then the 4 points determine a convex quadrilateral.

Case 3: $k = 3$. In this case, 3 of the points are the vertices of a triangle ABC containing the other 2 points. Draw a line through these 2 points. This line intersects 2 sides of the triangle, say AB and AC. Then B, C and the two vertices inside the triangle determine a convex quadrilateral.