Tate Resolutions for Products of Projective Spaces
(joint work with David A. Cox)

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Background for Tate Resolutions

Basic Notation

- V and W are dual vector spaces over k: $V = W^*$
- $\dim(V) = \dim(W) = N + 1$
- $E = \wedge^\bullet V$ is a graded exterior algebra
- $E_{-i} = \wedge^i V$ are graded parts (we assume $\deg(V) = -1$)

The Dualizing Module of E

- $\widehat{E} = \omega_E = \mathrm{Hom}_k(E, k)$ is a left E-module
- $\widehat{E}_i = \mathrm{Hom}_k(E_{-i}, k) = \mathrm{Hom}_k(\wedge^i V, k) = \wedge^i W$
- $\widehat{E}(p)$ is a graded E-module with $\widehat{E}(p)_q = \widehat{E}_{p+q}$
- $\widehat{E} \cong E(-N-1)$ (non-canonically)

Definition of Tate Resolution

Tate Resolution

- V and W are dual vector spaces over k: $V = W^*$ (dim $V = N + 1$)
- $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}^N = \mathbb{P}(W) = (W - \{0\})/k^*$
- Tate resolution is a bi-infinite exact sequence

$$T^\bullet(\mathcal{F}) : \cdots \to T^{-1}(\mathcal{F}) \to T^0(\mathcal{F}) \to T^1(\mathcal{F}) \to \cdots \to T^p(\mathcal{F}) \to \cdots$$

of free graded modules over exterior algebras $E = \wedge^\bullet V$.

Terms (Eisenbud, Fløystad and Schreyer, 2003, [EFS 03, ES 03])

$$T^p(\mathcal{F}) = \bigoplus_i \widehat{E}(i-p) \otimes_k H^i(\mathbb{P}(W), \mathcal{F}(p-i)),$$

where $\widehat{E} = \mathrm{Hom}_k(E, k) = \wedge^\bullet W$ as an E-module.

Why should we study Tate resolutions?

Tate resolution keeps a LOT of information

1. Beilinson-Gelfand-Gelfand (BGG) correspondence:
   $D^b(\mathbb{P}(W)) = Kom^\bullet(E - mod)$
2. Algebraic properties of coherent sheaves
   - Regularity of $\mathcal{F}$
   - Duality
   - Koszul cohomology (introduced by M. Green) $\longleftarrow$ **new**
3. Elimination theory
   - Resultants (A. Khetan, [Kh1, Kh2], D. Eisenbud, F.-O. Schreyer, [ES 03])
   - Hyperdeterminants

Known facts about the maps

1. If $i < j$, then the map

$$d^p_{i,j} : \widehat{E}(i-p) \otimes H^i(\mathcal{F}(p-i)) \to \widehat{E}(j-p-1) \otimes H^j(\mathcal{F}(p+1-j))$$

in $d^p = \oplus_{i,j} d^p_{i,j} : T^p \to T^{p+1}$ is zero.

2. The (i,i)-components of the map $d^p : T^p \to T^{p+1}$ are known explicitly:

$$\widehat{E}(i-p) \otimes H^i(\mathcal{F}(p-i)) \to \widehat{E}(i-p-1) \otimes H^i(\mathcal{F}(p+1-i))$$
$$f \otimes m \longmapsto \sum_i f\, e_i^* \otimes e_i\, m$$

where $\{e_i\}_{i=\overline{1,N}}$ is a basis of V; $\{e_i^*\}_{i=\overline{1,N}}$ is a basis of W, and correspond to

$$W \otimes H^i(\mathcal{F}(p-i)) \to H^i(\mathcal{F}(p+1-i)),$$

i.e., are the Koszul-type maps.

Known facts about the maps

3. For each differential

$$d^p : T^p(\mathcal{F}) \to T^{p+1}(\mathcal{F})$$

- $T^{\geq p}(\mathcal{F})$ is a minimal injective resolution of $\ker(d^p)$
- $T^{<p}(\mathcal{F})$ is a minimal projective resolution of $\ker(d^p)$

4. Recall that a coherent sheaf $\mathcal{F}$ is called m-regular if

$$H^i(\mathcal{F}(m-i)) = 0, \quad \text{for all } i > 0.$$

If $p \geq m = \mathrm{reg}(\mathcal{F})$, then

$$\cdots \to T^{m-2}(\mathcal{F}) \to T^{m-1}(\mathcal{F}) \to \widehat{E}(-m) \otimes H^0(\mathcal{F}(m)) \to \cdots.$$

Notation

- $X = \mathbb{P}^{\ell_1} \times \cdots \times \mathbb{P}^{\ell_r}$, $\dim(X) = \ell_1 + \cdots + \ell_r = \ell$
- $S = k[\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(r)}]$ – graded polynomial ring in r groups of variables
- $\mathbf{x}^{(i)} = (x_0^{(i)}, \ldots, x_{\ell_i}^{(i)})$, where for all $i = 1, \ldots, r$
  $\deg(x_0^{(i)}) = \cdots = \deg(x_{\ell_i}^{(i)}) = (0, \ldots, 1, \ldots, 0)$
- For $(n_1, \ldots, n_r) \in \mathbb{Z}^r$ denote the sheaf
  $$\mathcal{O}_X(n_1, \ldots, n_r) = p_1^* \mathcal{O}_{\mathbb{P}^{\ell_1}}(n_1) \otimes \cdots \otimes p_r^* \mathcal{O}_{\mathbb{P}^{\ell_r}}(n_r),$$
  where $p_j : \mathbb{P}^{\ell_1} \times \cdots \times \mathbb{P}^{\ell_r} \to \mathbb{P}^{\ell_j}$ is the projection.
- The subspace in S of polynomials in $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(r)}$ homogeneous of degrees $n_i \geq 0$ in each $\mathbf{x}^{(i)}$ is
  $$S_{n_1, \ldots, n_r} = H^0(X, \mathcal{O}_X(n_1, \ldots, n_r))$$

Notation

- Fix the degree vector $d = (d_1, \dots, d_r) \in \mathbb{Z}^r_{>0}$
- Let ν_d be the embedding
$$\nu_d : X = \mathbb{P}^{\ell_1} \times \cdots \times \mathbb{P}^{\ell_r} \longrightarrow \mathbb{P}(W), \quad W = S_{d_1,\dots,d_r}$$
  which is a combination of Veronese and Segre embeddings
- Consider the sheaf
$$\mathcal{F} = \nu_{d*}\mathcal{O}_X(m_1, \dots, m_r)$$
- Since
$$\mathcal{O}_{\mathbb{P}(W)}(1)|_{\nu_d(X)} = \nu_{d*}\mathcal{O}_X(d_1, \dots, d_r),$$
  we have:
$$H^i(\mathbb{P}(W), \mathcal{F}(j)) = H^i(X, \mathcal{O}_X(m_1 + jd_1, \dots, m_r + jd_r))$$
- Now the Tate resolution has the terms $T^p(\mathcal{F}) = \oplus_i T^p_i$:
$$T^p_i = \widehat{E}(i-p) \otimes H^i(X, \mathcal{O}_X(m_1 + (p-i)d_1, \dots, m_r + (p-i)d_r))$$

The Case of Veronese embedding of $\mathbb{P}^n$ (D. Cox)

- $X = \mathbb{P}^n$; $\mathcal{F} = \nu_{d*}\mathcal{O}_{\mathbb{P}^n}(\ell)$ for any $\ell \in \mathbb{Z}$, where
- $\nu_d : \mathbb{P}^n \to \mathbb{P}(W)$ is the d-fold Veronese embedding
- $W = S_d \subset S = k[x_0, \dots, x_n]$ polynomials of degree d
- Since $\mathcal{O}_{\mathbb{P}(W)}(1)|_{\nu_d(\mathbb{P}^n)} = \nu_{d*}\mathcal{O}_{\mathbb{P}^n}(d)$, we have
$$T^p(\mathcal{F}) = \widehat{E}(-p) \otimes S_{\ell+pd} \bigoplus \widehat{E}(n-p) \otimes S^*_{-n-1-(\ell+(p-n)d)}$$
- The map $T^p(\mathcal{F}) \to T^{p+1}(\mathcal{F})$ has the following form:

$$\begin{tikzcd}
\widehat{E}(n-p) \otimes S^*_{\rho-a} \arrow[r, "\alpha_p"] \arrow[dr, "\delta_p"] & \widehat{E}(n-p-1) \otimes S^*_{\rho-a-d} \\
\bigoplus & \bigoplus \\
\widehat{E}(-p) \otimes S_{a-d} \arrow[r, "\beta_p"] & \widehat{E}(-p-1) \otimes S_a,
\end{tikzcd}$$

where $a = \ell + (p+1)d$, $\rho = (n+1)(d-1)$

The Case of Veronese embedding of $\mathbb{P}^n$ (D. Cox)

- For $f \in k[x_0, \ldots, x_n]$ and $0 \leq j \leq n$ of degree d, define
$$\Delta_j(f) = \frac{f(y_0,\ldots,y_{j-1},x_j,x_{j+1},\ldots,x_n) - f(y_0,\ldots,y_{j-1},y_j,x_{j+1},\ldots,x_n)}{x_j - y_j}$$
- The Bezoutian of homogeneous polynomials $f_0, \ldots, f_n \in k[x_0, \ldots, x_n]$ of degree d is the determinant
$$\Delta = \det \Delta_j(f_i) = \sum_{|\alpha| \leq \rho} \Delta_\alpha(x) y^\alpha = \sum_{|\alpha| \leq \rho} \Delta_\alpha(x) \otimes x^\alpha$$
- The Bezoutian in degree $(\rho - a, a)$ gives a linear map
$$\bigwedge^{n+1} W = \bigwedge^{n+1} S_d \to S_{\rho-a} \otimes S_a,$$
which corresponds to an E-module homomorphism
$$B_p : \widehat{E}(n-p) \otimes S^*_{\rho-a} \to \widehat{E}(-p-1) \otimes S_a$$
- **Theorem.** (D. Cox, 2007, [Cox 07]) The map δ_p in $T^p(\mathcal{F}) \to T^{p+1}(\mathcal{F})$ is equal to $(-1)^p B_p$ defined by the Bezoutian.

The Case of Segre embedding of $\mathbb{P}^a \times \mathbb{P}^b$

- $X = \mathbb{P}^a \times \mathbb{P}^b$ – product of projective spaces
- $\mathcal{F} = \nu_* \mathcal{O}_X(k, \ell)$ for any $k, \ell \in \mathbb{Z}$, where
- $\nu : X \to \mathbb{P}(W)$ is the Segre embedding
- W is spanned by $x_i y_j$, $0 \le i \le a$, $0 \le j \le b$
- In particular,

$$\mathrm{reg}(\mathcal{F}) = \max\{-\min\{k, \ell\}, \min\{b-k, a-\ell\}\}$$

- $S = k[\mathbf{x}, \mathbf{y}] = k[x_0, \ldots, x_a; y_0, \ldots, y_b]$ – polynomial ring
- Grading: $\deg(x_i) = (1, 0)$, $\deg(y_j) = (0, 1)$
- Bi-homogeneous part of $S_{m,n} \subset S$ is spanned by $x^\alpha y^\beta$, $|\alpha| = m$, $|\beta| = n$

The Shape of the Resolution

Terms

The terms of the Tate resolution are

$$T^p(\mathcal{F}) = \oplus_i \widehat{E}(i-p) \otimes H^i(X, \mathcal{O}_X(k+p-i, \ell+p-i)),$$

where

$$H^i(X, \mathcal{O}_X(k+p-i, \ell+p-i)) = 0 \text{ for } i \notin \{0, a, b, a+b\}.$$

Types of the resolution

There are three types of the resolution of $\mathcal{F} = \nu_* \mathcal{O}_X(k, \ell)$ on $X = \mathbb{P}^a \times \mathbb{P}^b$:

I) $-a \leq k - \ell \leq b$

II) $k - \ell > b$

III) $k - \ell < -a$ (similar to type II))

Terms of the Resolution of Type I

Terms corresponding to Koszul maps (for all types)

Define the numbers:

$p^+ = \max\{-\min\{k,\ell\}, \min\{b-k, a-\ell\}\} = \mathrm{reg}(\mathcal{F})$,
$p^- = \min\{-\min\{k,\ell\}, \min\{b-k, a-\ell\}\} - 1$. Then

$$T^p(\mathcal{F}) = \begin{cases} \widehat{E}(-p) \otimes S_{k+p,\ell+p} & p \geq p^+ \\ \widehat{E}(a+b-p) \otimes S^*_{b-k-1-p,a-\ell-1-p} & p \leq p^-. \end{cases}$$

Terms corresponding to the non-Koszul maps of Type I resolution

Assume that $\mathcal{F}$ has Type I ($-a \leq k-\ell \leq b$).
Then $p^- = -\min\{k,\ell\} - 1$ and $p^+ = \min\{b-k, a-\ell\}$.
Furthermore, if $p^- < p < p^+$, then

$$T^p(\mathcal{F}) = \begin{array}{c} \widehat{E}(a+b-p) \otimes S^*_{b-k-1-p,a-\ell-1-p} \\ \bigoplus \\ \widehat{E}(-p) \otimes S_{k+p,\ell+p}. \end{array}$$

Maps in the resolution of Type I

Toric Jacobian

Given $f_0, \dots, f_{a+b} \in W = S_{1,1}$, where $f_j(x, y) = \sum_{i,k} a_{i,j,k}\, x_i y_j$,
the toric Jacobian is

$$J(f_0, \dots, f_{a+b}) = \frac{1}{x_0 y_b} \det \begin{pmatrix} f_0 & \cdots & f_{a+b} \\ \frac{\partial f_0}{\partial x_1} & \cdots & \frac{\partial f_{a+b}}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial f_0}{\partial x_a} & \cdots & \frac{\partial f_{a+b}}{\partial x_a} \\ \frac{\partial f_0}{\partial y_0} & \cdots & \frac{\partial f_{a+b}}{\partial y_0} \\ \vdots & & \vdots \\ \frac{\partial f_0}{\partial y_{b-1}} & \cdots & \frac{\partial f_{a+b}}{\partial y_{b-1}} \end{pmatrix} \in S_{b,a}.$$

We get a linear map $J : \bigwedge^{a+b+1} W \longrightarrow S_{b,a}$.

Maps in the resolution of Type I

The non-Koszul part of the differential $d^p : T^p(\mathcal{F}) \to T^{p+1}(\mathcal{F})$ looks like

$$\widehat{E}(a+b-p) \otimes S^*_{b-k-1-p,a-\ell-1-p} \to \widehat{E}(-p-1) \otimes S_{k+p+1,\ell+p+1}$$

and is induced by the map

$$\delta_1 : \wedge^{a+b+1} W \to S_{b-k-1-p,a-\ell-1-p} \otimes S_{k+p+1,\ell+p+1}.$$

The change of variables

$$J \mapsto J(X_i + x_i, Y_j + y_j) \in k[\mathbf{X}, \mathbf{Y}, \mathbf{x}, \mathbf{y}] \cong S \otimes S$$

in the toric Jacobian extends the map J to

$$J = \oplus_{\alpha,\beta} J_{\alpha,\beta}, \quad J_{\alpha,\beta} : \bigwedge^{a+b+1} W \longrightarrow S_{b-\alpha,a-\beta} \otimes S_{\alpha,\beta}.$$

Theorem. The map δ_1 can be chosen to be $(-1)^p J_{k+p+1,\ell+p+1}$.

Maps in the resolution of Type II

Terms corresponding to the non-Koszul maps of Type II resolution

Assume that $\mathcal{F}$ has Type 2 ($k - \ell > b$).
Then $p^- = b - k - 1$ and $p^+ = -\ell$. Furthermore, if $p^- < p < p^+$, then

$$T^p(\mathcal{F}) = \widehat{E}(b - p) \otimes S_{k+p-b,0} \otimes S^*_{0,-\ell-p-1}.$$

Differentials:

The differential in $T^{p^-}(\mathcal{F}) \to T^{p^-+1}(\mathcal{F})$ looks like

$$d^- : \widehat{E}(a+1+k) \otimes S^*_{0,a+k-\ell-b} \to \widehat{E}(k) \otimes S_{0,0} \otimes S^*_{0,k-\ell-b-1}.$$

The differential in $T^{p^+-1}(\mathcal{F}) \to T^{p^+}(\mathcal{F})$ looks like

$$d^+ : \widehat{E}(b+1+\ell) \otimes S_{k-\ell-b-1,0} \otimes S^*_{0,0} \to \widehat{E}(\ell) \otimes S_{k-\ell,0}.$$

Maps in the resolution of Type II

For given $f_0, \ldots, f_{a+b} \in W = S_{1,1}$ write $f_i = \sum_j A_{ij} x_j$, $A_{ij} \in S_{0,1}$ and define the map

$$\gamma_\alpha : \bigwedge^{a+1} W \xrightarrow{M} S_{0,a+1} \longrightarrow S^*_{0,\alpha} \otimes S_{0,a+1+\alpha},$$

where $M(f_0, \ldots, f_a) = \det(A_{ij}) \in S_{0,a+1}$, and

$$S_{0,a+1} \longrightarrow S^*_{0,\alpha} \otimes S_{0,a+1+\alpha}$$

is the comultiplication map. This induces (by abuse of notation) the maps:

$$\gamma_\alpha : \widehat{E}(a+1+k) \otimes S_{0,\alpha} \longrightarrow \widehat{E}(k) \otimes S_{0,a+1+\alpha}$$
$$\gamma^*_\alpha : \widehat{E}(a+1+k) \otimes S^*_{0,a+1+\alpha} \longrightarrow \widehat{E}(k) \otimes S^*_{0,\alpha}.$$

Theorem. The non-Koszul differentials in the Tate resolution of Type II can be chosen to be $d^- = \gamma^*_{k-\ell-b-1}$ and $d^+ = \gamma_{k-\ell-b-1}$.

Example (on maps of Type I)

Let $\nu : X = \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}(W) = \mathbb{P}^5$, $\mathcal{F} = \nu_* \mathcal{O}_X(0,1)$.

$$\begin{tikzcd}
\cdots \arrow[r] & \widehat{E}(4) \otimes S^*_{1,1} \arrow[r] \arrow[drr, "J_{0,1}"] & \widehat{E}(3) \otimes S^*_{0,0} \arrow[drr, "-J_{1,2}"] & & \\
& & \bigoplus \; 0 \; \bigoplus \; 0 \; \bigoplus & & \\
& & & \widehat{E}(0) \otimes S_{0,1} \arrow[r] & \widehat{E}(-1) \otimes S_{1,2} \arrow[r] & \cdots
\end{tikzcd}$$

Example (on maps of Type II)

Let $\nu : X = \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}(W) = \mathbb{P}^5$, $\mathcal{F} = \nu_* \mathcal{O}_X(3,0)$.
The only nonzero diagonal maps appear in $T^{-3}(\mathcal{F}) \to T^{-2}(\mathcal{F})$:

$$\begin{tikzcd} \cdots \arrow[r] & \widehat{E}(6) \otimes S^*_{0,4} \arrow[dr, "d^-"] & \\ & & \widehat{E}(3) \otimes S_{0,0} \otimes S^*_{0,1} \arrow[r] & \cdots \end{tikzcd}$$

(at cohomological levels 3 and 1) and in $T^{-1}(\mathcal{F}) \to T^{0}(\mathcal{F})$:

$$\begin{tikzcd} \cdots \arrow[r] & \widehat{E}(2) \otimes S_{1,0} \otimes S^*_{0,0} \arrow[dr, "d^+"] & \\ & & \widehat{E}(0) \otimes S_{3,0} \arrow[r] & \cdots \end{tikzcd}$$

(at cohomological levels 1 and 0).

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