

Witt Vectors' Action  
on K-groups of Monoid Rings

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Rutgers AMS meeting

Oct. 6-7, 2007

# Crash course in (higher) K-theory

2.

$R$  a (commutative) ring

$\text{IP}(R)$  fin. gen. projective  $R$ -modules

$K_0(R) =$  Free abelian group on isomor. classes  $(P)$  where  $P \in \text{IP}(R)$

$$K(P \oplus Q) - (P) - (Q) \}$$

Fact  $K_0(R) = \mathbb{Z} \iff \forall P \in \text{IP}(R)$   
is stably free

Theorem (Grothendieck)  $R$  regular.

$$K_0(R) = K_0(R[t]) = K_0(R[t_1, \dots, t_n])$$

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$$K_i(R) = \overline{\lim}_{i(+1)} (X_R)$$

$X_R$  a huge topological space

constructed out of  $GL_n(R)$   
( $\text{IP}(R)$ )

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$K_i(R)$  very subtle invariants,  
hard to compute

$K_1(R)$  and  $K_2(R)$  can be defined  
 in pure algebraic terms

$$K_1(R) = \frac{GL(R)}{[GL(R), GL(R)]}$$

$$U(R) = GL_1(R) \subset GL_2(R) \subset \dots \subset GL_n(R)$$

$$\ast \mapsto \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$$

$\times$  scheme (variety)

Iso. classes of vect. bundles on  $\times$

$$K_0(\times) = \frac{\{(V) - (V') - (V'') : 0 \rightarrow V' \rightarrow V + V'' \rightarrow 0\}}{\{(V) - (V') - (V'') : 0 \rightarrow V' \rightarrow V + V'' \rightarrow 0\}}$$

$K_i(X) = \overline{\bigoplus_{i+1}^{\infty}}$  (topological space  
constructed out of  
the cat. of vector  
bundles/ $X$ )

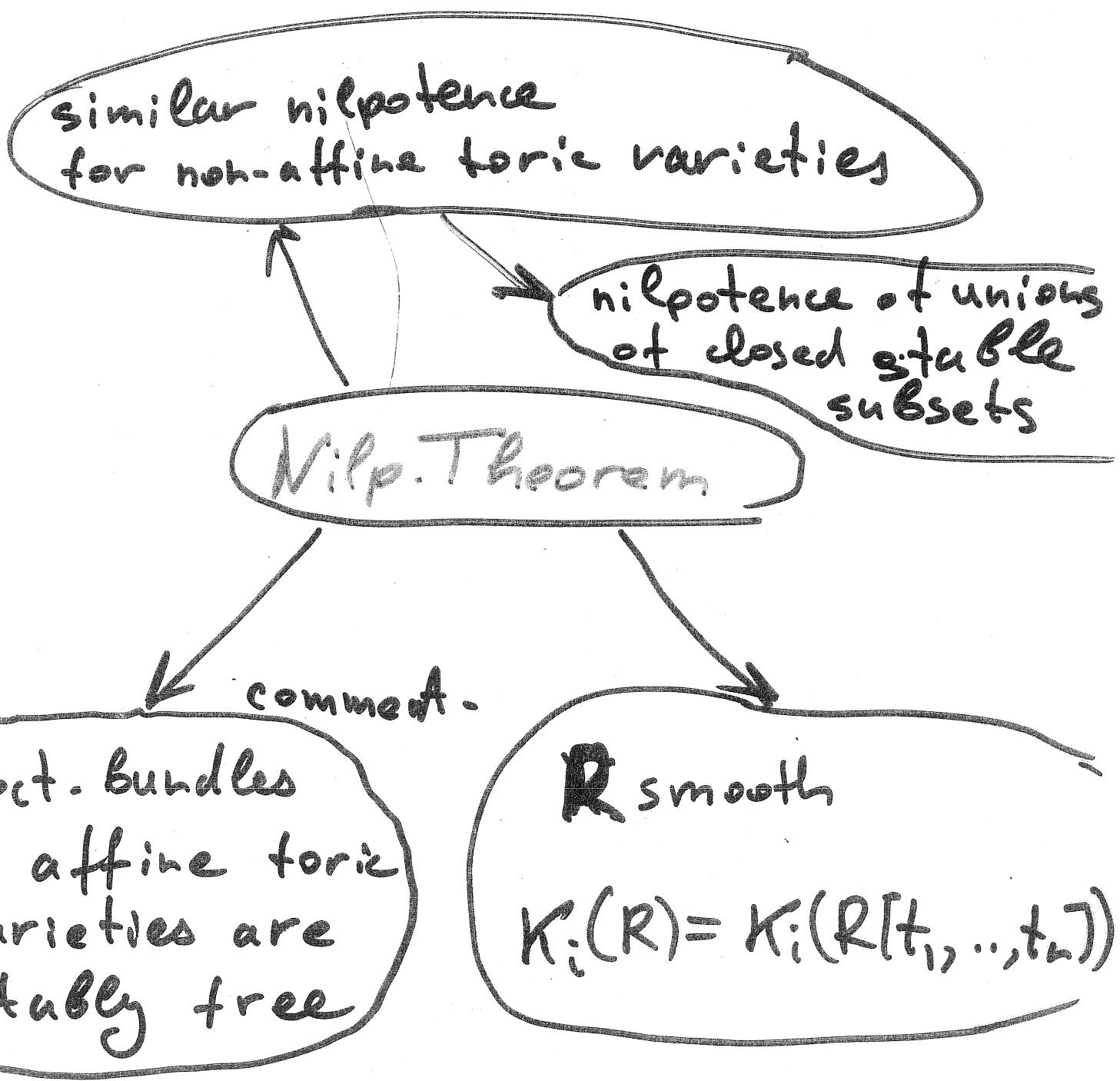
Theorem  $M$  an additive sub-monoid of a rational vector space (e.g. affine normal positive), without nontrivial invertibles.  $R$  a regular ring, containing  $\mathbb{Q}$ .  $c \geq 2$  a natural number,  $i \geq 0$ ,  $x \in K_i(R[M])$ . Then  $(c^i)_*(x) \in K_i(R)$  iff

Here:  $c_* : K_i(R[CM]) \rightarrow K_i(R[M])$   
 induced by  $M \mapsto M^c$ ,  $m \in M$ ,  
 and  $K_i(R) = K_i(R[CM])$  functorially

Observe:  $(\langle i \rangle)_* = (\langle \_ \rangle^i)_*$

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## Motivation



Submitted by: [unclear] comments

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Nilpotence cannot be strengthened:

$$K_i(R) = K_i(R[M]), \quad i \geq 1$$

M an affine monoid

$i \leq 1$  brought?

$$M = \sum_{+}^r, \quad \text{so } R[M] = R[t_1, \dots, t_n]$$

For  $i=0$  the similar implication  
is false

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History: when  $R$  is a char. 0  
field: Gubeladze, 2003 (published  
in 2005), in 2007 Haesemeyer –  
Cortiñas – Walker – Weibel:  
new proof (considerably shorter)

# Crash Course in Big Witt

## Vectors

$R$  commutative ring

$W(R)$  the ring of Big Witt Vectors, whose underlying additive structure is the multiplicative group of formal power series of the form

$$1 + a_1 t + a_2 t^2 + \dots \in R[[t]]$$

Fact

$W(\mathbb{Z}_p)$  = the ring of  $p$ -adic integers

Fact  $\theta \in R$

$W(R) = R^N$  (nontrivial ghost)

Bloch-Stienstra-Weibel  
operations

$$A = R \oplus A_1 \oplus A_2 \oplus \dots$$

$W(R)$  acts on  $K_i(A) / K_i(R)$

(non-trivial even for  $i=0$ )

Fact When  $R$  is a characteristic 0 field

$K_i(A) / K_i(R)$  is  $R$ -vector space

$$R \xrightarrow{\text{diag}} R^N = W(R)$$

Ghost copy of  $R$ .

A word on the proof of  
Mal'tsev-Thm

$R$  a field of char 0.

$M$  an affine positive monoid  
(i.e. an additive submonoid  
of  $\mathbb{Z}^n$  without nontrivial  
invertibles)

There exists a grading

$$R[M] = R \oplus A_1 \oplus A_2 \oplus \dots$$

The induced  $W(R)$ -action on  
 $K_i(R[M]) / K_i(R)$  is related  
to the action

$$c_* : K_i(R[M]) \rightarrow K_i(R[M])$$

$$c \geq 2$$

The relationship in terms of

One essentially uses Continas  
proof of 'KABI'-conjecture  
(in both approaches : G, HCWW)

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The globalization of the coefficient  
ring, from a field of char. 0  
to a regular ring  $\mathbb{Q} \subset R$ , also  
uses  $W(R)$ -actions

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Future Directions

M affine monoid  
R char. 0 field

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$$R[M] = R \oplus A_1 \oplus A_2 \oplus \dots$$

a grading (M consists of homogeneous elements)

Conjecture

$$K_i(R[M]) / K_i(R)$$

generated  $W(R)$ -module

(arithmetic  $R$ ?)

Fact

Conjecture  $\Rightarrow$  Nilp. Thm.

(for  $R$  a field)

char = 0

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Question  $R$  char.  $p > 0$

field. Is  $K_i(R[U]) / K_i(R)$   
a fin. gen.  $W(R)$ -module?

What is the relationship with  
K-theoretical nilpotence in  
this case?

Question  $A = R \oplus A_1 \oplus A_2 \oplus \dots$

fin. gen.  $R$ -algebra,  $R$  a field.

Is  $K_i(A) / K_i(R)$  a fin. gen.

$W(R)$ -module? Nontrivial  
already for  $i=0$

Essentially non-toric cases of  
this question considered by Hasselblad