A ring structure on intersection cohomology of hypertoric varieties

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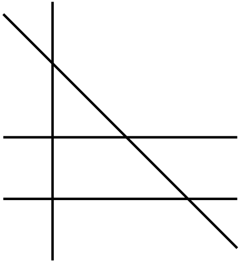
Outline

To a rational hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^d$, associate a *Hypertoric variety* $\mathfrak{M}_{\mathcal{H}}$.

- $\dim_{\mathbb{C}} \mathfrak{M}_{\mathcal{H}} = 2d$, torus $T = (\mathbb{C}^*)^d$ acts



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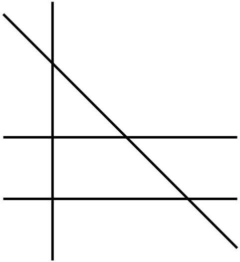

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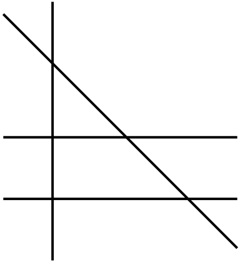

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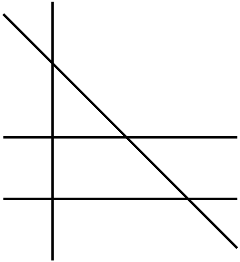
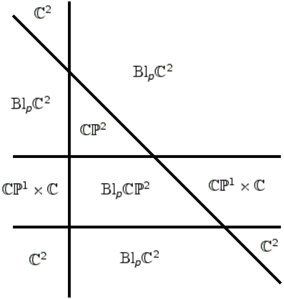
- Never compact

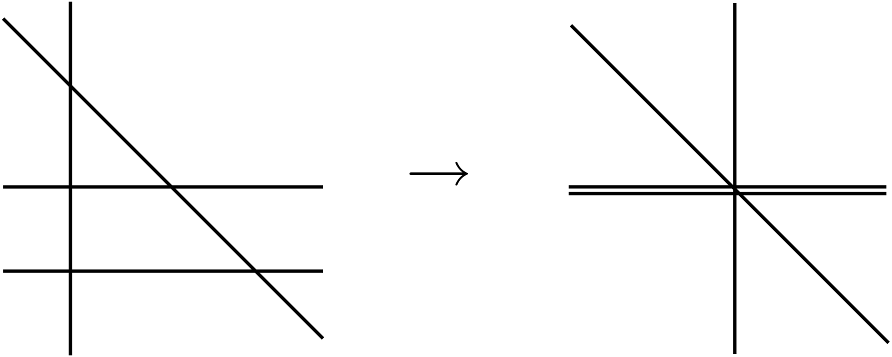


The toric varieties X_P whose moment polyhedra are the chambers of $\mathcal{H}$ are Lagrangian subvarieties of $\mathfrak{M}_{\mathcal{H}}$.



If $\mathfrak{M}_{\mathcal{H}}$ is smooth, then every X_P is smooth, and $\mathfrak{M}_{\mathcal{H}} = \bigcup_P T^*X_P$.

If $\mathcal{H}$ is central, then $\mathfrak{M}_{\mathcal{H}}$ is affine. If $\widetilde{\mathcal{H}}$ is a simplification of $\mathcal{H}$, there is a map $\mathfrak{M}_{\widetilde{\mathcal{H}}} \to \mathfrak{M}_{\mathcal{H}}$ which is an (orbifold) resolution of singularities.

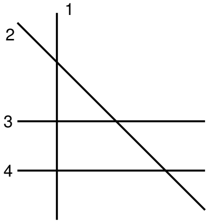


Equivariant cohomology

If $\mathcal{H} = \{H_1, \ldots, H_n\}$ is simple, $\exists$ canonical ring isomorphism

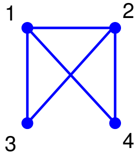
$$H^\bullet_T(\mathfrak{M}_\mathcal{H}) = \mathbb{R}[e_1, \ldots, e_n]/\langle \prod_{i\in S} e_i \mid \bigcap_{i\in S} H_i = \varnothing \rangle.$$

This is the face ring $\mathbb{R}[\Delta_\mathcal{H}]$ of the matroid complex of $\mathcal{H}$.



$$H^\bullet_T(\mathfrak{M}_\mathcal{H}) = \mathbb{R}[e_1, e_2, e_3, e_4]\,/\langle e_1 e_2 e_3, e_1 e_2 e_4, e_3 e_4 \rangle$$

$\Delta_\mathcal{H}$:



IH Betti numbers

Theorem (Proudfoot-Webster '04)

If $\mathcal{H}$ is central, then there is an isomorphism

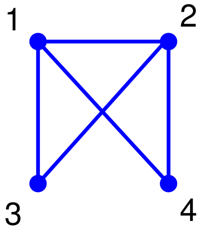
$$IH^\bullet_T(\mathfrak{M}_\mathcal{H}) \cong \mathbb{R}[\Delta^{bc}_\mathcal{H}]$$

of $H^\bullet_T(pt)$-modules.

$\Delta^{bc}_\mathcal{H}$ = "broken circuit complex"
= simplices of $\Delta_\mathcal{H}$ containing no broken circuit.
circuit = minimal non-face C of $\Delta_\mathcal{H}$
broken circuit = $C \smallsetminus \min(C)$.

This isomorphism is not canonical. $\Delta^{bc}_\mathcal{H}$ depends on the choice of ordering of the hyperplanes, although its Betti numbers do not.

Example



Circuits: $\{1,2,3\}$, $\{1,2,4\}$, $\{3,4\}$

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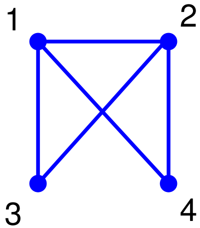

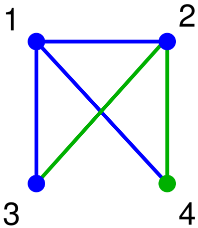

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Broken circuits: $\{2,3\}$, $\{2,4\}$, $\{4\}$

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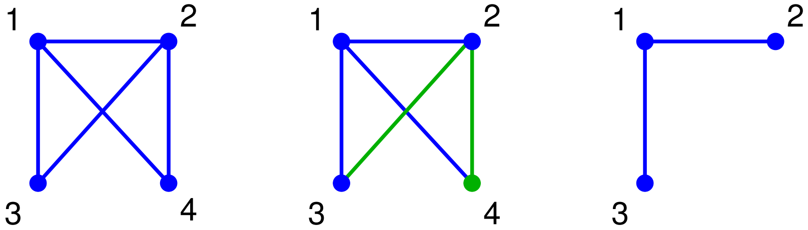

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$\Delta_{\mathcal{H}}^{bc} = \{\{1,2\},\{1,3\},\{1\},\{2\},\{3\},\varnothing\}.$

Proudfoot and Speyer constructed a Cohen-Macaualay ring $R(\mathcal{H})$ which degenerates to $\mathbb{R}[\Delta_{\mathcal{H}}^{bc}]$ for any choice of ordering:

$$R(\mathcal{H}) = \mathbb{R}[e_1, \ldots, e_n]/\langle \sum_{i \in C} a_i \prod_{j \in C \smallsetminus i} e_j = 0 \rangle$$

where C runs over all circuits, and $\sum_{i \in C} a_i v_i = 0$ is a linear dependence among the normal vectors v_i to the hyperplanes H_i. In particular $R(\mathcal{H})$ has the same graded dimension as $\mathbb{R}[\Delta_{\mathcal{H}}^{bc}]$.

Question

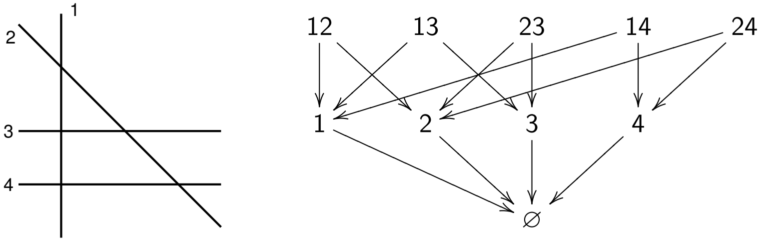
Is there a *canonical* identification $R(\mathcal{H}) \cong IH_T^\bullet(\mathfrak{M}_{\mathcal{H}})$?

Minimal extension sheaves

Minimal extension sheaves on fans (Barthel-Brasselet-Fieseler-Kaup, Bressler-Lunts) give a canonical functorial computation of IH_T of toric varieties. We adapt this formalism to arrangements and hypertoric varieties...

Let $L_{\mathcal{H}}$ = the lattice of flats of $\mathcal{H}$. If $\mathcal{H}$ is simple, this is just the matroid complex $\Delta_{\mathcal{H}}$.

$E \leq F$ means E lies in *fewer* hyperplanes — E is larger as a subspace of $\mathbb{R}^d$.



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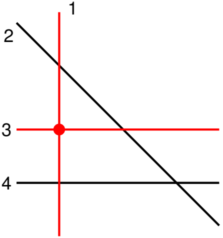
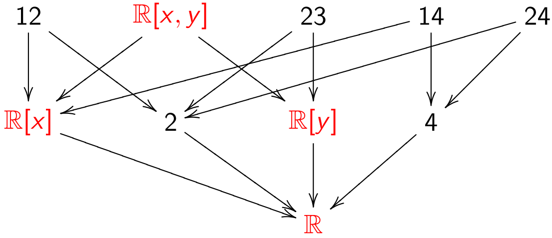
For any flat F, define $\mathcal{A}(F) = \mathrm{Sym}(N_F)$, where N_F is the normal space to F in $\mathbb{R}^d$.





The quotient maps $\mathcal{A}(F) \to \mathcal{A}(E)$ when $E \leq F$ make $\mathcal{H}$ into a sheaf of graded rings on $L_{\mathcal{H}}$, with the order topology.

A sheaf $\mathcal{M}$ on $L_{\mathcal{H}}$ is an $\mathcal{A}$-module if $\mathcal{M}(F)$ is a graded $\mathcal{A}(F)$-module for each flat F, and the restriction maps are maps of modules.

Definition

An $\mathcal{A}$-module $\mathcal{L}$ is a **minimal extension sheaf** if

1. $\mathcal{L}(\varnothing) = \mathcal{A}(\varnothing) = \mathbb{R}$
2. $\mathcal{L}(F)$ is a free $\mathcal{A}(F)$-module for all F
3. $\mathcal{L}$ is flabby — sections extend upward
4. $\mathcal{L}$ is minimal with respect to 1, 2, and 3.

Theorem (B.-Proudfoot)

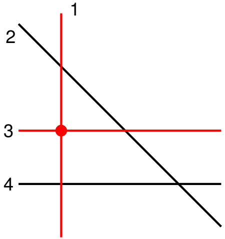
Any two minimal extension sheaves on $L_{\mathcal{H}}$ are canonically isomorphic, up to a scalar.

If $\mathcal{H}$ is rational, then there is a canonical isomorphism

$$\mathcal{L}(L_{\mathcal{H}}) \cong IH^{\bullet}_{T}(\mathfrak{M}_{\mathcal{H}}).$$

Example

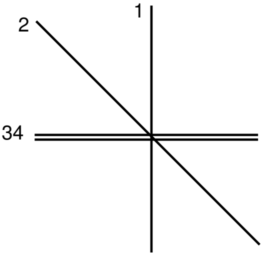
If $\mathcal{H}$ is simple, then $\mathcal{A}$ itself is a minimal extension sheaf. Its global sections are the face ring $\mathbb{R}[\Delta_{\mathcal{H}}]$.



$$\begin{tikzcd}
12 \arrow[d] \arrow[dr] & \mathbb{R}[x,y] \arrow[dl] \arrow[dr] & 23 \arrow[dl] \arrow[d] & 14 \arrow[dlll] \arrow[d] & 24 \arrow[dlll] \arrow[dl] \\
\mathbb{R}[x] \arrow[drr] & 2 \arrow[dr] & \mathbb{R}[y] \arrow[d] & 4 \arrow[dl] & \\
 & & \mathbb{R} & &
\end{tikzcd}$$

Another example

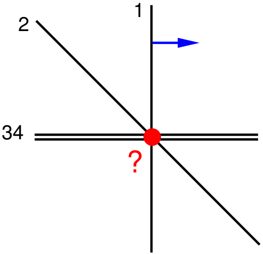
For the central version of our arrangement, $\mathcal{A}$ is not flabby:



$$\begin{tikzcd}
 & 1234 \arrow[dl] \arrow[d] \arrow[dr] & \\
1 \arrow[dr] & 2 \arrow[d] & 34 \arrow[dl] \\
 & \varnothing &
\end{tikzcd}$$

Another example

For the central version of our arrangement, $\mathcal{A}$ is not flabby: sections cannot be extended to the point.

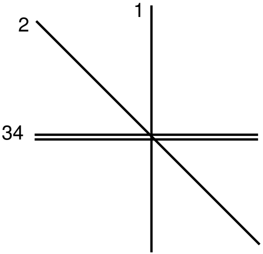


$$\begin{tikzcd}
 & A \arrow[dl] \arrow[d] \arrow[dr] & \\
A/\langle y\rangle \arrow[dr] & A/\langle x-y\rangle \arrow[d] & A/\langle x\rangle \arrow[dl] \\
 & A/\langle x,y\rangle &
\end{tikzcd}$$

$$A = \mathbb{R}[x,y]$$

Another example

Adding an extra generator at the point, we get a flabby sheaf:



$$\begin{tikzcd}
& A \oplus \xi A \arrow[dl] \arrow[d] \arrow[dr] & \\
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Localization

The sheaves $\mathcal{A}$ and $\mathcal{L}$ come from localizing equivariant cohomology and intersection cohomology

$\mathfrak{M}_{\mathcal{H}}$ has a stratification $\bigcup_F S_F$ indexed by $L_{\mathcal{H}}$. The T-stabilizer is the same for any point $p \in S_F$, and

$$H^\bullet_T(Tp) \cong \mathrm{Sym}((\mathfrak{t}_F)^*) = \mathcal{A}(F).$$

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$\mathcal{L}(F)$ is the equivariant IH "stalk" along Tp. If p degenerates from a large stratum S_E to a small one S_F, this induces a map

$$\mathcal{L}(F) \to \mathcal{L}(E)$$

which is the restriction map for the sheaf $\mathcal{L}$.

A normal slice to the stratum S_F in $\mathfrak{M}_{\mathcal{H}}$ is isomorphic to the affine hypertoric variety $\mathfrak{M}_{\mathcal{H}_F}$ defined by the *localization* of $\mathcal{H}$ at F: the central arrangement obtained by restricting to hyperplanes in F and slicing.

Thus we have an isomorphism

$$\mathcal{L}(F) \cong IH^\bullet_T(\mathfrak{M}_{\mathcal{H}_F}) \cong \mathbb{R}[\Delta^{bc}_{\mathcal{H}_F}] \cong R(\mathcal{H}_F).$$

For flats $E \leq F$, we can define a ring homomorphism $R(\mathcal{H}_F) \to R(\mathcal{H}_E)$ by setting the variables e_i, $i \in F \setminus E$ to zero.

With these maps, $F \mapsto R(\mathcal{H}_F)$ defines an $\mathcal{A}$-module $\mathcal{R}$.

Theorem (B.–Proudfoot)

$\mathcal{R}$ is a minimal extension sheaf.

Corollary

If $\mathcal{H}$ is a rational central arrangement, there is a canonical isomorphism

$$R(\mathcal{H}) = \mathcal{R}(L_{\mathcal{H}}) \cong IH^{\bullet}_{T}(\mathfrak{M}_{\mathcal{H}}).$$

In particular, $IH^{\bullet}_{T}(\mathfrak{M}_{\mathcal{H}})$ carries a canonical ring structure.

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How can we understand this ring structure?

Theorem (B.–Proudfoot)

If $\mathcal{H}$ is unimodular, then the equivariant IC sheaf $\mathbf{IC}_T(\mathfrak{M}_{\mathcal{H}})$ can be made into a ring object in the equivariant derived category $D^b_T(\mathcal{H})$ by a multiplication map

$$\mathbf{IC}_T(\mathfrak{M}_{\mathcal{H}}) \otimes \mathbf{IC}_T(\mathfrak{M}_{\mathcal{H}}) \to \mathbf{IC}_T(\mathfrak{M}_{\mathcal{H}}).$$

This ring structure is unique, and it induces our ring structure on $IH^\bullet_T(\mathfrak{M}_{\mathcal{H}})$.

This implies that the ring structure respects a number of other functorial maps besides the restrictions in the sheaf $\mathcal{R}$. For instance, restriction to the open stratum $S_\varnothing$ gives a ring homomorphism

$$R(\mathcal{H}) = IH^\bullet_T(\mathfrak{M}_{\mathcal{H}}) \to H^\bullet_T(S_\varnothing).$$

Why is unimodularity needed?

The unimodularity hypothesis is puzzling. The sheaf $\mathcal{R}$ makes sense, gives a minimal extension sheaf, and has the "right" Betti numbers even if $\mathcal{H}$ is not unimodular, or even not rational.

But there is an isomorphism of rings:

$$H^\bullet_T(S_\varnothing) \cong \mathbb{R}[e_1, \ldots, e_n]/\langle e_1^2, \ldots, e_n^2\rangle + \langle \sum_{i \in C} \operatorname{sgn}(a_i) \prod_{j \in C \smallsetminus i} e_j = 0\rangle.$$

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If $\mathcal{H}$ is unimodular, then $\operatorname{sgn}(a_i) = a_i$, so this matches up with

$$R(\mathcal{H}) = \mathbb{R}[e_1, \ldots, e_n]/\langle \sum_{i \in C} a_i \prod_{j \in C \smallsetminus i} e_j = 0\rangle.$$

Wild speculation

Could there be some sort of “orbifold corrections” when $\mathcal{H}$ is rational but not unimodular which make a topological description of our ring structure possible?