INTRODUCTION TO SCHEMES - HOMEWORK 8

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As before, you do no need to do all these exercises for credit, but are expected to choose at least three that are challenging to you at your level.

If you think you have discovered a mistake in one of these exercises, please email me as soon as possible.

(1) Let $X = \mathrm{Spec}(R)$ be an affine scheme, where $R \cong K[x_1, \dots, x_n]/I$. Let P be a closed point of X. Show that X is singular at P if and only if the Jacobian matrix has rank less than $n - \dim(X)$.

(2) Let $X = \mathrm{Spec}(\mathbb{C}[x, y, z, w]/\langle xw - yz \rangle)$, and let D be the prime divisor obtained by setting $x = y = 0$. Write ν_D for the valuation corresponding to D. What is $\nu_D(x^i y^j z^k w^l)$ for different values of i, j, k, l?

(3) Let $X = \mathrm{Proj}(\mathbb{C}[x, y, z]/\langle x^3 + 3y^3 + xyz + 2z^3 \rangle)$. Let D be the prime divisor obtained by setting $x + y = 0$, and let D' be the prime divisor obtained by setting $y + 3z = 0$. Show that D and D' are linearly equivalent.

(4) Let $X = \mathrm{Proj}(\mathbb{C}[x, y, z, w]/\langle xw - yz \rangle)$, and let $D = \mathrm{Proj}(\mathbb{C}[x, y, z, w]/\langle x, y \rangle$. Give a Cartier divisor description $\{(U_i, f_i)\}$ for D (if possible!)

(5) Show that $\mathcal{O}(d)$ is a globally generated line bundle on $\mathbb{P}^n$. Show that if we take $\{s_i\}$ to be a basis for the global sections of $\mathcal{O}(d)$ then the corresponding morphism $\mathbb{P}^n \to \mathbb{P}^N$ is the d-uple Veronese embedding (google if you don't know what this is, or Harris *Algebraic Geometry*).

(6) Hartshorne 6.2.

(7) Hartshorne 6.6 (this will involve looking at the relevant example in section 6 first).

(8) Hartshorne 7.2

(9) Hartshorne 7.3