INTRODUCTION TO SCHEMES - HOMEWORK 5

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As before, you do no need to do all these exercises for credit, but are expected to choose at least three that are challenging to you at your level.

If you think you have discovered a mistake in one of these exercises, please email me as soon as possible.

(1) The p-adic valuation on $\mathbb{Q}$ is given by $\mathrm{val}(p^n a/b) = n$, where $a, b \in \mathbb{Z}$ with p not dividing a, b. Let $R = \{q \in \mathbb{Q} : \mathrm{val}(q) \geq 0\}$.
  (a) Describe $\mathrm{Spec}(R)$.
  (b) Fix $p = 2$. Let $\phi\colon R \to R[x]/\langle x^2 - 4\rangle$ be the inclusion map. What are the fibres of corresponding map $\phi\colon \mathrm{Spec}(R[x]/\langle x^2 - 4\rangle) \to \mathrm{Spec}(R)$?

(2) Let $\phi\colon \mathrm{Spec}(\mathbb{Z}[x]/\langle 2x-3\rangle) \to \mathrm{Spec}(\mathbb{Z})$ be induced from the inclusion $\mathbb{Z} \to \mathbb{Z}[x]/\langle 2x-3\rangle$. Compute the fibers of ϕ at each point of $\mathrm{Spec}(\mathbb{Z})$.

(3) Let $X = \mathrm{Spec}(\mathbb{C}[x,y]/\langle x(x-1), y(y-1), xy\rangle$, $Y = \mathrm{Spec}(\mathbb{C}[x,y]/\langle x(x-1), y(y-1), (x-1)(y-1)\rangle$, and $Z = \mathrm{Spec}(\mathbb{C}[x])$. Let $\phi\colon X \to Z$ be given by the inclusion of $\mathbb{C}[x]$ into $\mathbb{C}[x,y]/\langle x(x-1), y(y-1), xy\rangle$, and $\psi\colon Y \to Z$ be given by the inclusion of $\mathbb{C}[x]$ into $\mathbb{C}[x,y]/\langle x(x-1), y(y-1), (x-1)(y-1)\rangle$. What is $W = X \times_Z Y$? In particular, check that the topological space of W is finite. How many points does it contain?

(4) Show that if $\phi\colon R \to S$ is a surjective homomorphism of graded rings (so homogeneous elements are taken to homogeneous elements), then there is a corresponding closed embedding $\mathrm{Proj}(S) \to \mathrm{Proj}(R)$.

(5) Vakil 6.4E (taking Veronese subrings does not change Proj).

(6) Let $R = \mathbb{C}[x^2, xy, y^2, xz, yz, z^2]$, $S = \mathbb{C}[x^2, xy, y^2, xz, yz]$. Show that $\mathrm{Proj}(R) \cong \mathbb{P}^2$, and $\mathrm{Proj}(S)$ is the blow-up of $\mathbb{P}^2$ at a point (see Ch 1 Hartshorne, or many other references if you do not know what the blow-up is). This means that we have a natural map $\mathrm{Proj}(S) \to \mathrm{Proj}(R)$, even thought we have an inclusion of rings $S \to R$.

(7) Hartshorne 3.15 (geometrically irreducible)

(8) Hartshorne II.4.1 (finite morphisms are proper)

(9) Hartshorne II.4.6 (proper morphisms of affine varieties over k are finite).
(10) Hartshorne II.4.8. (preservation of properties of morphisms)
(11) Vakil 9.4B (properties preserved by base change)
(12) (Not to be handed in). Read the proof of Vakil 10.1.8.
(13) (Not to be handed in). Look up Yoneda's lemma in category theory. Think about it.