INTRODUCTION TO SCHEMES - HOMEWORK 3

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As before, you do no need to do all these exercises for credit, but are expected to choose at least three that are challenging to you at your level.

If you think you have discovered a mistake in one of these exercises, please email me as soon as possible.

(1) For each of the following ring homomorphisms $\phi\colon R \to S$, decide if the map $\phi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is finite type, finite, or neither.
   (a) $\phi\colon \mathbb{C}[x] \to \mathbb{C}[x, y, z]/\langle x^2 + y^2 + z^2\rangle$ given by $\phi(x) = x$.
   (b) $\phi\colon \mathbb{C}[x, y] \to \mathbb{C}[x, y, z]/\langle x^2 + y^2 + z^2\rangle$ given by $\phi(x) = x$, $\phi(y) = y$.
   (c) $\phi\colon \mathbb{Z} \to \mathbb{Z}[x]/\langle 2x - 3\rangle$.
   (d) $\phi\colon \mathbb{C}[x] \to \mathbb{C}[x, y]/\langle xy - 1\rangle$, where $\phi(x) = x$.
   (e) $\phi\colon \mathbb{C}(x) \to \mathbb{C}((x))$, where $\mathbb{C}((x))$ is the field of Laurent series, and the homomorphism takes a rational function to its Taylor expansion.

(2) Show that if $(X, \mathcal{O})$ is a scheme, and U, V are disjoint open subsets in X, then $\mathcal{O}(U \cup V) \cong \mathcal{O}(U) \times \mathcal{O}(V)$.

(3) Show that an affine scheme is quasi-compact. (This has been done implicitly in lectures, and used in lectures, so the better prepared shouldn't count this as one of their three).

(4) Show that $\mathrm{Spec}(R)$ is reduced if and only if R has no nilpotent elements.

(5) Let $\phi\colon X \to Y$ be a morphism of schemes. Show that the condition on an open affine $U = \mathrm{Spec}(B)$ that "$\phi^{-1}(U)$ can be covered by open affines $\mathrm{Spec}(A)_i$ where each A_i is a finitely generated B algebra" satisfies the conditions of the Affine Communication Lemma. Conclude Hartshorne 3.1

(6) Hartshorne 3.5

(7) Let X be an integral scheme. Show that there is a point $\zeta \in X$ with closure all of X. This is called the generic point. Show that the stalk at ζ is a field; this is called the function field of X. What is $K(X)$ when X is an affine variety?

(8) Hartshorne 3.8

(9) Hartshorne 3.14
(10) Hartshorne 3.13 (for some of the properties you will need to wait until a future lecture or read around).