INTRODUCTION TO SCHEMES - HOMEWORK 3

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As before, you do no need to do all these exercises for credit, but are expected to choose at least three that are challenging to you at your level.

If you think you have discovered a mistake in one of these exercises, please email me as soon as possible.

(1) Let $R = \mathbb{C}[x]/\langle x^3 \rangle$, and $S = \mathbb{C}[x]/\langle x^2 \rangle$. Describe the morphism $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ corresponding to the ring homomorphism $\phi \colon R \to S$ given by $\phi(x) = x$. You should describe $\mathrm{Spec}(S)$, $\mathrm{Spec}(R)$, the continuous map of topological spaces, and the morphism of sheaves $\phi^{\#} \colon \mathcal{O}_{\mathrm{Spec}(R)} \to \mathcal{O}_{\mathrm{Spec}(S)}$.

(2) Let $R = \mathbb{C}[t]_{\langle t \rangle}$ and $S = \mathbb{C}[t]_{\langle t \rangle}[x]/\langle (x-t)(x-2t) \rangle$. Let $\phi \colon R \to S$ be the ring homomorphism that takes t to t. Repeat the previous exercise for this morphism.

(3) Give an example of a morphism of ringed spaces where every stalk is a local ring that is not a morphism of locally ringed spaces.

(4) Let $S = \mathbb{Z}[x, y, z]$. Let $X = \mathrm{Spec}(S)$, and $Y = D(x) = \mathrm{Spec}(S_x)$. Let $e_1 = x + y + z$, $e_2 = xy + xz + yz$, $e_3 = xyz$, $p_1 = x+y+z = e_1$, $p_2 = x^2+y^2+z^2$ and $p_3 = x^3+y^3+z^3$, Show that $Y = D(e_1) \cup D(e_2) \cup D(e_3)$, where $D(e_i)$ here is the basic open in Y (i.e. $e_i \in S_x$). Show that $D(p_1) \cup D(p_2) \cup D(p_3) \neq Y$. (For hints and more like this, see *Teaching the geometry of schemes* by Smith and Sturmfels).

(5) Do some of the Hartshorne exercises from HW2 (now that we have the definitions).

(6) Hartshorne 2.5.

(7) Hartshorne 2.18.

(8) Eisenbud-Harris Exercise I.25 (p22)

(9) Vakil Exercise 3.2R.