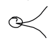
Last time:

- quasicoherent sheaves
  locally $\tilde{M}$ for M an R-module.
- coherent sheaves
- line bundles – locally free of rank 1.

<u>Today</u>: Divisors/line bundles

<u>Defn</u> Let (R, m, k) be a local ring. ($k = R/m$)
R is <u>regular</u> if $\dim_k m/m^2 = \dim R$ (as v.s. ; $\dim R$ = Krull dim)
(always have $\geq$)

eg $R = \left(k[x,y] / (x^2 - y^3) \right)_{\langle x,y \rangle}$



R is not a regular local ring, since $\dim_k m/m^2 = 2$ (basis x, y) but $\dim(R) = 1$

<u>Defn</u> A scheme X is singular at a pt x if the local ring $\mathcal{O}_x$ is not a regular local ring

<u>Defn</u> A scheme X is regular in codim one if every local ring $\mathcal{O}_x$ of dim one is regular.
(eg $X = \mathrm{Spec}(R)$ (R domain), P point
$\dim R_P = \max\text{ length} \{ P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P \mid P_i \text{ prime in } R \}$)
$\dim R_P = 1$ iff $\mathrm{Spec}(R/P)$ is codim 1 in $\mathrm{Spec}(R)$

eg If X is nonsingular at every point p.

eg If X is Noetherian and <u>normal</u>
(local rings are integrally closed domains)
(Serre: normal is R1 + S2.

<u>Defn</u> Let X be a Noetherian integral separated scheme that is regular in codim 1.
A <u>prime divisor</u> on X is a closed integral subscheme of codimension one.
A <u>Weil divisor</u> is a formal linear combination of prime divisors.
$$D = \sum_{\text{finite}} a_i D_i \quad a_i \in \mathbb{Z}.$$

This forms a group $\mathrm{Div}(X)$.
If all $a_i \geq 0$ then D is <u>effective</u>.

Lemma

Defn Let X be an integral scheme. Then there is a unique point $x \in X$ with $\overline{\{x\}} = X$.

eg $X = \mathbb{A}^n$ $x = \langle 0 \rangle$

The stalk at x $\mathcal{O}_x$ is called the function field $K(X)$ of X.

eg $k[x_1, \ldots, x_n]_{\langle 0 \rangle} = k(x_1, \ldots, x_n)$

$K(X)$ is the field of fractions of R for any open affine $U = \operatorname{Spec}(R)$ on X.

Prime divisors give divisorial valuations on $K(X)$.

Let Y be a prime divisor, Let U be an open affine of X intersecting Y. Then $Y \cap U = \operatorname{Spec}(R/P)$ for a prime P of R.

The local ring R_P is a regular local ring, of dim 1, so is a DVR (discrete valuation ring) with field of fractions $K(X)$. Equivalently, there is $\pi \in R_P$ s.t. $P = \langle \pi \rangle$ & every element can be written uniquely as $u\pi^n$ for some unit u & $n \in \mathbb{Z}$.

(eg $\mathbb{C}[t]_{\langle t \rangle} \ni \frac{f}{g}$ g has a constant term.

$= \frac{f' t^n}{g}$

$\mathbb{Z}_{\langle p \rangle} = \left\{ \frac{a}{b} \mid p \nmid b \right\}$

Define $v_Y(u\pi^l) = l$ for $u\pi^l \in K(X)$.

eg $X = \mathbb{A}^n$, $Y = V(x_1)$

This has local ring $k[x_1, \ldots, x_n]_{\langle x_1 \rangle}$ with valuation $v_Y(x_1^k f) = k$ if $x_1 \nmid f$.

$K(X) = k(x_1, \ldots, x_n)$.

Lemma Let X be a noetherian integral separated scheme that is regular in codim 1. Fix $f \in K(X)$. Then $v_Y(f) = 0$ for all but finitely many prime divisors

eg $X = \operatorname{Spec}(\mathbb{Z})$, $K(X) = \mathbb{Q}$

Prime divisors on $X \leftrightarrow$ primes

$f = \frac{a}{b}$ $v_p(f) = n$ if $\frac{a}{b} = p^n \frac{a'}{b'}$ $p \nmid a', b'$.

eg $X = \operatorname{Spec}(\mathcal{O}_K)$ ring of integers in a number field.

Getting new batteries.

Defn Given $f \in K(X)$, we define the divisor of f to be

$$\operatorname{div}(f) = \sum_{Y \text{ prime divisor}} v_Y(f)\, Y \in \operatorname{Div}(X).$$

These are called principal divisors.

Defn Two Weil divisors D, D' are linearly equivalent if $D = D' + \operatorname{div}(f)$ for some $f \in K(X)$.

The (Divisor) class group of X is $\operatorname{Div}(X)/\text{principal divisors}$.

Cartier divisors

Idea: "locally principal divisor".

Precisely: $\mathcal{K}$ is the sheaf on X obtained by sheafifying the total quotient rings $\mathcal{K}(U)$ of $\mathcal{O}_X(U)$ for every open set U.

$Q(R) = R[S^{-1}]$
S set of nzd of R.

$\mathcal{K}^*$ is the sheaf of groups consisting of the invertible elements.

A Cartier divisor is a global section of $\mathcal{K}^*/\mathcal{O}_X^*$ ← invertible elements in $\mathcal{O}_X$.

ie locally (U, f) where $f \in \mathcal{K}^*(U)$,

& $\forall U_1, U_2$ $f_1|_{U_1 \cap U_2} = f_2|_{U_1 \cap U_2}$ mod $\mathcal{O}_X^*$

ie $\frac{f_1}{f_2} \in \mathcal{O}_X^*(U_1 \cap U_2)$

Can describe a Cartier divisor by giving an open cover $X = \bigcup U_i$ & $\forall i$ $f_i \in \mathcal{K}^*(U_i)$ satisfying

Defn A Cartier divisor is principal if it is in the image of $\mathcal{K}^*(X) \to \mathcal{K}^*/\mathcal{O}^*$

Cartier divisors $\mathcal{K}^*/\mathcal{O}_X^*(X)$

$\updownarrow$

$X = \bigcup U_i \quad (U_i, f_i) \quad f_i \in \mathcal{K}^*(U_i)$

Notes:

1) If X is integral, separated, Noetherian, regular in codim 1, (so Weil divisors work well)

Then $\mathcal{K}^*$ is the constant sheaf $K(X)$.

Given an open cover $X = \bigcup U_i$ & a Cartier divisor $\{(U_i, f_i)\}$ we get a Weil divisor $D = \sum_Y \nu_Y(f_i) Y$ ← any i with $Y \cap U_i \neq \emptyset$

(does not depend on the choice of i since if $Y \cap U_i \neq \emptyset$, $Y \cap U_j \neq \emptyset$, then $\frac{f_i}{f_j} \in \mathcal{O}_X^*$ so $\nu_Y\left(\frac{f_i}{f_j}\right) = 0$ so $\nu_Y(f_i) = \nu_Y(f_j)$. (using $\nu_Y\left(\frac{a}{b}\right) = \nu_Y(a) - \nu_Y(b)$.)

This takes principal Cartier ← f_i of $\forall i$ divisors to principal Weil divisors.

So we get a group homomorphism

$$\text{Cartier divisors}/\text{principal divisors} \longrightarrow \text{Div}(X)/\text{principal divisors}.$$

Prop (6.11 Hartshorne).

If X is regular (all local rings are regular) then this homomorphism is an isomorphism.

2) It is not always surjective.

eg $X = \text{Spec}\left(\frac{\mathbb{C}[x,y,z,w]}{\langle xw - yz\rangle}\right)$

affine cone over $\mathbb{P}^1 \times \mathbb{P}^1$.

$D = V(x,y) \subseteq X$ (codimension one in X.)

$\text{Spec}\left(\frac{\mathbb{C}[x,y,z,w]}{\langle x,y\rangle}\right) \longrightarrow X$

This is a prime Weil divisor, but not a Cartier divisor.

ie $\nexists$ $X = \bigcup U_i$ where $D|_{U_i} = V(f_i)$ for some f_i.

Check: $z \neq 0$. $\left(\frac{\mathbb{C}[x,y,z,w]}{\langle xw - yz\rangle}\right)_z = \left(\frac{\mathbb{C}[x,y,z,w]}{\langle y - \frac{xw}{z}\rangle}\right)_z = \mathbb{C}[x,z,w]_z$

$D|_{z \neq 0} = V(x,y)$.

$D = V\left(x, \frac{xw}{z}\right) = V(x)$ ← principal.

(Not a surprise since $X|_{z\neq 0}$ is nonsingular)

$X = \mathrm{Spec}\left(\dfrac{\mathbb{C}[x,y,z,w]}{\langle xw - yz\rangle}\right)$.

$D = V(x,y)$.

Suppose $X = \bigcup U_i$ is open cover, and $(0,0,0,0) \in U_1$.

We claim that $D|_{U_1}$ is not principal.

If it were, $D|_{D(f)}$ would be principal for some basic open contained in U_1.

ie $\langle x,y\rangle R_f$ would be a principal ideal for $R = \dfrac{\mathbb{C}[x,y,z,w]}{\langle xw - yz\rangle}$.

But then $\langle x,y\rangle R_p$ would be principal for $p = \langle x,y,z,w\rangle$.

This would imply that pR_p has at most 3 generators so $\dim_k \frac{p}{p^2} \leq 3 = \dim R_p$, which contradicts that X is singular at p.

3) A Cartier divisor gives rise to a line bundle (locally free sheaf of rank 1).

$D = \{(U_i, f_i)\} \longmapsto \mathcal{L}(D)$

$\mathcal{L}(D)(U_i) = \mathcal{O}_X f_i^{-1} \leq \mathcal{K}(U_i)$

eg $U_i = \mathrm{Spec}(R_i)$, $R_i f_i^{-1} \leq Q(R_i)$

eg $X = \mathrm{Spec}(\mathbb{Z})$ $D = \langle 3\rangle$ $(\mathrm{Spec}(\mathbb{Z}), 3)$

$\mathbb{Z}\frac{1}{3} = \left\{\frac{a}{3} \,\middle|\, a \in \mathbb{Z}\right\} \simeq \mathbb{Z}$ as a $\mathbb{Z}$-module

Well-defined:

f_i^{-1} & f_j^{-1} generate the same $\mathcal{O}_X(U_1 \cap U_2)$-module as $\frac{f_1}{f_2}$ is invertible in $\mathcal{O}_X(U_1 \cap U_2)$

ie map Cartier divisors $\longrightarrow$ Line bundles

Prop: $\mathcal{L}(D_1) \simeq \mathcal{L}(D_2)$ as invertible sheaves (ignore embedding into $\mathcal{K}$)

if & only if $D_1 \sim D_2$

linear equivalent.

- $\mathcal{L}(D_1 + D_2) = \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)$

$f_1 \quad f_2$

$f_1 f_2 \qquad \frac{1}{f_1 f_2}$

$\Longrightarrow$ $\dfrac{\text{Cartier divisors}}{\text{principal divisors}} \longrightarrow \operatorname{Pic}(X)$

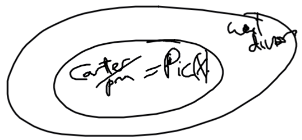
(group of line bundles up to isomorphism. $\otimes$, $\mathcal{O}_X$ is identity.

Prop If X is integral then this is an isomorphism.

ie for nice X

X nonsingular

Weil/princp = Cartier/princ = $\operatorname{Pic}(X)$.



Maps to projective space

Defn A line bundle $\mathcal{L}$ is globally generated if $\forall x \in X$ $\exists s \in \mathcal{L}(X)$ s.t. $s_x \in \mathcal{L}_x$ generates $\mathcal{L}_x$ as an $\mathcal{O}_x$-module (ie $s_x \notin m_x \mathcal{L}_x$)
stalk

$\mathcal{L}$ is generated by global sections $s_0, \ldots, s_n$ if $\forall x$ the s_{ix} generate $\mathcal{L}_x$.

Let X be a scheme over $\mathrm{Spec}(K)$ $(\exists X \to \mathrm{Spec}(K))$

Given $s_0, \ldots, s_n$ generating $\mathcal{L}$ on X. we define

$$\varphi : X \longrightarrow \mathbb{P}^n$$

1) $X_i = \{ p \in X \mid (s_i)_p \notin m_p \mathcal{L}_p \}$

Then $\bigcup_{i=0}^{n} X_i$ is an open cover of X.

Define

$$\varphi_i^\# : K\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right] \longrightarrow \mathcal{O}_X(X_i)$$

$$\frac{x_j}{x_i} \longmapsto \frac{s_j}{s_i} \in \mathcal{O}_{X_i}(X_i)$$

not in $m_p \mathcal{L}_x$

This induces

$$X_i \longmapsto U_i = D(X_i) \subseteq \mathbb{P}^n$$

These morphism glue to get

$$X \longrightarrow \mathbb{P}^n$$

2) Alternative description:

Let S be the vector space spanned by $s_0, \ldots, s_n$.

$$\mathbb{P}^n = \mathbb{P}(\underbrace{S^\vee}_{\text{dual } k\text{-v.s.}})$$

$$x \longmapsto \{ \underbrace{\{ s \in S \mid s \in m_x \mathcal{L}_x \}}_{\text{hyperplane in } S} \} \in \mathbb{P}(S^\vee)$$

Defn The line bundle $\mathcal{L}$ is very ample if φ is a closed embedding. It is ample if $\mathcal{L}^{\otimes n}$ is very ample for some $n > 0$.

eg $\mathcal{L} = \mathcal{O}(2)$ on $\mathbb{P}^n$

φ is the second Veronese embedding