Last time:

functor of points

$h_X = \mathrm{Mor}(-, X)$

Moduli spaces

eg Grassmannian.

<u>Today</u>: quasicoherent sheaves

<u>Recall</u>: A sheaf of $\mathcal{O}_X$-modules is a sheaf $\mathcal{F}$ s.t $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module for all U, & the module structure is compatible with restriction

<u>Quasicoherent sheaves</u>

Let R be a ring & M an R-module.

Let $\tilde{M}$ be the sheaf on $\mathrm{Spec}(R)$ corresponding to the sheaf on the base given by $\tilde{M}(D(f)) = M_f$. $\forall f \in R$

(recall: $\mathcal{O}_{\mathrm{Spec}(R)}(D(f)) = R_f$, and M_f is an R_f-module.)

Ex: This is a sheaf on the base!

Explicitly,

$$\tilde{M}(U) = \Big\{ s : U \to \coprod_{p \in U} M_p : \forall p,\ s(p) \in M_p, \text{ and } \exists V \subseteq U, p \in V, m \in M, f \in R, f \notin Q\ \forall Q \in V, \text{ s.t } s(Q) = \tfrac{m}{f}\ \forall Q \in V \Big\}$$

Easy egs

- $R = K$ field, $M = K^n$

$\mathrm{Spec}(R) = \{\langle 0 \rangle\}$

$\tilde{M}(\emptyset) = 0 \quad \tilde{M}(\mathrm{Spec}(R)) = K^n$

- $R = \mathbb{C}[[t]]$ $\quad \mathrm{Spec}(R) = \{\langle 0 \rangle, \langle t \rangle\}$

$M = \mathbb{C}[[t]]/\langle t \rangle \simeq \mathbb{C}$

Open sets: $\{\emptyset, \mathrm{Spec}(R), \underset{\overset{\shortparallel}{D(t)}}{\langle 0 \rangle}\}$

$\tilde{M}(\emptyset) = 0 \quad \tilde{M}(\mathrm{Spec}(R)) = M$

$\tilde{M}(\langle 0 \rangle) = M_t = $ R_t module

$R_t = \mathbb{C}((t))$ ← Laurent series

Prop

Let M be an R-module & let $X = \mathrm{Spec}(R)$.

a) $\tilde{M}$ is an ~~an~~ sheaf of $\mathcal{O}_X$-modules

b) The stalk of $\tilde{M}$ at P is M_P

c) The map $M \to \tilde{M}$ gives an exact full faithful functor from the category of R-modules to the category of $\mathcal{O}_X$-modules.
(ie $\varphi: M \to N$ induces $\varphi: \tilde{M} \to \tilde{N}$

d) The $\sim$ construction commutes with tensor product & direct sum

$$\widetilde{M \otimes_R N} = \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N}$$

$$\widetilde{\left(\bigoplus M_i\right)} = \bigoplus \tilde{M_i}$$

Defn Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on X is quasi coherent if X can be covered by open affines $\overset{U=}{}\mathrm{Spec}(R)$ with $\mathcal{F}|_U = \tilde{M}$ for some R-module M.

eg $\mathcal{O}_X$ is a quasi-coherent $\mathcal{O}_X$-module.

Theorem (Vakil 13.3.1)
If P is the property of affine opens $\mathrm{Spec}(R)$ on X that $\mathcal{F}|_{\mathrm{Spec}(R)} = \tilde{M}$ for some R-module M, then P satisfies the properties of the affine communication lemma
(If $\mathrm{Spec}(R)$ has P, $\mathrm{Spec}(R_f)$ does,
If $\langle f_1, \ldots, f_3 \rangle = R$, & $\mathrm{Spec}(R_{f_i})$ has P $\forall i$, $\mathrm{Spec}(R)$ does)

Cor, If $\mathcal{F}$ is a quasi coherent sheaf of $\mathcal{O}_X$-modules then for every open affine $\mathrm{Spec}(R)$ on X, $\mathcal{F}|_{\mathrm{Spec}(R)} \simeq \tilde{M}$ for some R-module M.

Coherent sheaves

Note: If M is an R-module, then we have 3 finiteness conditions:

i) M is finitely generated
$\exists R^p \twoheadrightarrow M$.

ii) M is finitely presented
$\exists R^q \to R^p \to M \to 0$

iii) coherent: $\exists R^p \twoheadrightarrow M$, & for all $\varphi: R^q \to M$ $\ker(\varphi)$ is finitely gen.

iii) $\Rightarrow$ ii) $\Rightarrow$ i) & for Noetherian rings they all coincide.

Defn A quasicoherent sheaf $\mathcal{F}$ is coherent if for every affine open $\mathrm{Spec}(R)$ $\mathcal{F}|_{\mathrm{Spec}(R)} \simeq \tilde{M}$ for a coherent R-module M.

Coherent sheaves on X form an abelian category.

Important examples of quasicoherent sheaves

① **Defn** A sheaf $\mathcal{I}$ is an ideal sheaf if $\mathcal{I}(U)$ is an ideal in $\mathcal{O}_X(U)$ for all U.

Defn Let Y be a closed subscheme of X & let $i: Y \to X$ be the inclusion morphism. The ideal sheaf of Y, $\mathcal{I}_Y$, is the kernel of $i^\#: \mathcal{O}_X \to i_* \mathcal{O}_Y$

$i_* \mathcal{O}_Y(U) = \mathcal{O}_Y(i^{-1}(U))$

Prop Let Y be a closed subscheme of a scheme X. The ideal sheaf $\mathcal{I}_Y$ on X is a quasicoherent sheaf of ideals. Conversely, any quasicoherent sheaf of ideals on X is $\mathcal{I}_Y$ for a unique closed subscheme Y of X.

Sketch of proof

1)
- $\mathcal{O}_Y$ is a quasicoherent $\mathcal{O}_Y$-module.
- pushforward of quasicoherent modules along quasicompact & separated morphisms are quasicoherent
- kernels of morphisms of quasicoherent are quasicoherent

$\mathcal{I}_Y = \ker(\mathcal{O}_X \to i_* \mathcal{O}_Y)$

2) If $\mathcal{I}$ is a quasicoherent sheaf of ideals on X, let Y be the support of $\mathcal{O}_X/\mathcal{I}$

$(Y = \{p \in X \mid (\mathcal{O}_X/\mathcal{I})_p \neq 0\})$

We claim that $(Y, \mathcal{O}_X/\mathcal{I}|_Y)$ is a closed subscheme of X. Check locally.

$\mathcal{I}|_{\mathrm{Spec}(R)} = \tilde{I}$ for an ideal I of R.

$Y \cap \mathrm{Spec}(R)$ is the closed subscheme of $\mathrm{Spec}(R)$ given by I.

(Back to eg

$R = \mathbb{C}[[t]]$, $M = \mathbb{C}[[t]]/(t)$

$\widetilde{M}(\langle 0 \rangle) = M_t$

$= \mathbb{C}[[t]]_{(t)} / (t)_{(t)}$

$= 0$

$\phi \to 0$

$\langle 0 \rangle \to 0$

$(\langle 0 \rangle, (t)) \to M \cong \mathbb{C}$

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② $\mathcal{O}(n)$ on $\mathbb{P}^n$

Recall: If R is a graded ring, with $m = \bigoplus_{i>0} R_i$

$\mathrm{Proj}(R)$ is the scheme with underlying set $\{P \mid P$ is homog. not containing $m\}$

$\mathcal{O}_{\mathrm{Proj}(R)}(D(f)) = \mathrm{Spec}((R_f)_0)$

eg $R = K[x_0, \ldots x_n]$

A R-module M is graded if $M = \bigoplus_{i \in \mathbb{Z}} M_i$ &

$R_i M_j \subseteq M_{i+j}$.

Given a graded R-module M we construct a quasicoherent sheaf $\widetilde{M}$ on $\mathrm{Proj}(R)$ as follows.

On $D(f) = \mathrm{Spec}((R_f)_0)$

set $\widetilde{M}(D(f)) = (M_f)_0$

$\left(\deg\left(\frac{m}{f^k}\right) = \deg(m) - k\deg(f)\right)$

Check: This gives a sheaf on the base.

We have $\widetilde{M}|_{D(f)} = \widetilde{(M_f)_0}$

The stalk: $\widetilde{M}_p = (M_p)_0 =: M_{(p)}$

As befor, explicitly

$\widetilde{M}(U) = \{s : U \to \coprod_{p \in U} (M_p)_0 \mid$

$\forall p,\ s(p) \in (M_p)_0$, $\forall p \exists V \subseteq U, p \in V$, $m \in M, f \in R$ st $\deg(m) = \deg(f)$

$f \notin Q \ \forall\ Q \in V$,

$s(Q) = \frac{m}{f} \in (M_Q)_0 \ \forall Q \in V\}$

Since $\mathrm{Proj}(R)$ is covered by $D(f)$, $\widetilde{M}$ is quasi-coherent

Important special case.

Given a graded module M we define the twisted module $M(n)$ by
$M(n)_i = M_{n+i}$ (same group, action, different grading.)

eg $R = M = \mathbb{C}[x,y]$

$\mathbb{C}[x,y](2)$ has $\deg(1) = -2$
$\deg(x) = -1$
$\deg(x^2) = 0$

$\mathbb{C}[x,y](-3)$ has
$\deg(1) = 3$
$\deg(x) = 4$
$\deg(xy) = 5$

Defn Let $X = \mathrm{Proj}(R)$,
For any $n \in \mathbb{Z}$, define
$\mathcal{O}_X(n) = \widetilde{R(n)}$

For a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, we have $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$

eg $R = K[x_0, x_n]$ K a field
$\mathcal{O}(1)$ on $\mathrm{Proj}(R) = \mathbb{P}^n_K$
is the sheaf $\widetilde{R(1)}$

On the affine open $D(x_i)$ this is $\left(R(1)_{x_i}\right)_0 = ?$

$\left(R_{x_i}\right)_0 = K\left[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right]$

$\left(R(1)_{x_i}\right)_0 = K\left[\frac{x_0}{x_i}, \frac{x_n}{x_i}\right]\{x_0, x_1, x_n\}$

The global sections of $\mathcal{O}(1)$ $(\mathcal{O}(1)(\mathbb{P}^n))$
are $K\{x_0, x_n\}$
span of

Similarly, for $\mathcal{O}(2)$,
$\left(R(2)_{x_i}\right)_0 = K\left[\frac{x_0}{x_i}, \frac{x_n}{x_i}\right]\{x_0^2, x_0x_1, \dots x_n^2\}$

In general, $\mathcal{O}(d)$ for $d \geq 0$ has global sections
$\longleftrightarrow R_d$

$\mathcal{O}(-1)$ $R(-1)$ 1 has deg 1
x_i has deg 2
x_i^2 has deg 3

$\left(R(-1)_{x_i}\right)_0 = K\left[\frac{x_0}{x_i}, \frac{x_n}{x_i}\right]\left\{\frac{1}{x_i}\right\}$
$\frac{1}{x_i}$ has deg $\deg(1) - \deg(x_i \text{ in } R)$
$= 1 - 1 = 0$

(Recall, in M_f

$$\deg\left(\frac{m}{f}\right) = \underset{\text{in } M}{\deg(m)} - \underset{\text{in } R}{\deg(f)}$$

so for $R(-1)$, $\deg\left(\frac{f}{g}\right)$

$$= \deg_{R(-1)}(f) - \deg_R(g).$$

Thus $\mathcal{O}(-1)(\mathbb{P}^n) = 0$

<u>Prop</u> Let R be a graded ring that is generated as R_0-alg by R_1.
Let $X = \operatorname{Proj}(R)$.
For any graded module M,

$$\widetilde{M(n)} = \tilde{M}(n) \overset{\text{''}}{} \tilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$$

In particular $\mathcal{O}_X(n) \otimes \mathcal{O}_X(m) \simeq \mathcal{O}_X(n+m)$

ie $\{\mathcal{O}_X(n) \mid n \in \mathbb{Z}\}$ forms a group under tensor product)

<u>Recall:</u> A sheaf $\mathcal{F}$ on X is locally free if it has an open cover by open sets U for which $\mathcal{F}|_U \simeq \bigoplus \mathcal{O}_X$.

A line bundle on X is a locally free sheaf of rank 1

ie $\mathcal{F}|_U \simeq \mathcal{O}_X$

This "is" the sheaf of sections of a topological line bundle.

$$\begin{array}{ll} Y & \\ \pi \downarrow & \pi^{-1}(p) \text{ a "line"} \\ X & \text{section: } s: X \to Y \\ & \text{st } \pi \circ s = \mathrm{id} \end{array}$$

<u>Ex:</u> Think about why this (sheaf of sections) should be free of rank 1 if $X = \operatorname{Spec}(R)$, & $Y = X \times \mathbb{A}^1$.