Last time:

Properties of schemes

- connected
- irreducible
- reduced
- integral
- (locally) Noetherian.

Properties of morphisms

- (locally) finite type
- finite
- affine.

Open embedding

Closed embedding

Fibre product

Recall: The fibre product in a category is given by the property:

If $\varphi : X \to Z$
$\quad\psi : Y \to Z$

then the fibre product $X \times_Z Y$ is

$$\begin{tikzcd} W \arrow[rrd, bend left] \arrow[rdd, bend right] \arrow[rd, dashed, "\exists!"] & & \\ & X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d, "\psi"] \\ & X \arrow[r, "\varphi"] & Z \end{tikzcd}$$

s.t. if
$$\begin{tikzcd} W \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & Z \end{tikzcd}$$

eg When the category is Sets $X \times_Z Y = \{(a,b) \in X \times Y \mid \varphi(a) = \psi(b)\}$,

Products don't always exist, but they do in the category of schemes!

First case: X, Y, Z, W all affine. $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(S)$, $Z = \mathrm{Spec}(T)$, $W = \mathrm{Spec}(A)$.

$$\begin{tikzcd} W \arrow[rr] \arrow[dd] \arrow[d, dashed, "\exists!"] & & Y \arrow[dd] \\ ? \arrow[rru] \arrow[d, dashed] & & \\ X \arrow[rr] & & Z \end{tikzcd}
\qquad
\begin{tikzcd} A & & S \arrow[ll, "\pi_2"'] \arrow[dl] \\ & R \otimes_T S \arrow[ul] & \\ R \arrow[uu, bend left, "\pi_1"] \arrow[ur] & & T \arrow[ll, "\varphi"] \arrow[uu, "\psi"'] \end{tikzcd}$$

$R \otimes_T S \to A$ $\quad r \otimes s \mapsto \pi_1(r)\pi_2(s)$

ie $X \times_Z Y = \mathrm{Spec}(R \otimes_T S)$.

General case: This glues (see eg Hartshorne / Vakil)

eg $\mathbb{A}^n_k = \mathrm{Spec}(k[x_1, \ldots, x_n])$

$$\mathbb{A}^n_k \times_{\mathrm{Spec}(k)} \mathbb{A}^m_k = \mathrm{Spec}(k[x_1, \ldots x_n] \otimes_k k[x_1, \ldots x_m])$$
$$= \mathrm{Spec}(k[x_1, \ldots x_{m+n}])$$
$$= \mathbb{A}^{n+m}$$

eg $\mathbb{A}^1_k \times_k \mathbb{A}^1_k = \mathbb{A}^2_k$

Topology is not the product topology!

Defn The residue field of a point p on a scheme X is $\mathcal{O}_p / m_p$ ← maximal ideal.

$$\mathcal{O}_p / m_p = \kappa(p)$$

eg $\mathbb{A}^2$, $(a,b) \leftrightarrow \langle x-a, y-b \rangle = p$.

$\mathcal{O}_p = K[x,y]_{\langle x-a, y-b \rangle}$ $m_p = \langle x-a, y-b \rangle$

$\kappa(p) = K$

If $\varphi : X \to Y$ is a morphism of schemes, then the fibre of φ at $y \in Y$ is

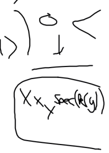
$$X \times_Y \operatorname{Spec}(k(y))$$

where $\operatorname{Spec}(k(y)) \to Y$ comes from

$\mathcal{O}(U) \to \mathcal{O}_y \to k(y)$ where U is an open containing y

eg $\operatorname{Spec}\left(\dfrac{\mathbb{C}[x,y]}{\langle y^2 - x^3 + 2x - 1 \rangle} \right)$

$\longrightarrow \operatorname{Spec}(\mathbb{C}[x])$



Fibre over $\langle x-1 \rangle$

$\kappa(\langle x-1 \rangle) = \mathbb{C}[x]_{\langle x-1 \rangle} / \langle x-1 \rangle \simeq \mathbb{C}$

$$\frac{\mathbb{C}[x,y]}{y^2 - x^3 + 2x - 1} \otimes_{\mathbb{C}[x]} \mathbb{C} \simeq \frac{\mathbb{C}[y]}{(y^2)}$$

Base change:

If $\varphi : X \to Y$ & $\psi : Z \to Y$, then $X \times_Y Z \to Z$ is the base change of φ

eg $Y = \operatorname{Spec}(K)$, $Z = \operatorname{Spec}(L)$

$K \subseteq L$.

Separated

Recall: A topological space X is Hausdorff if $\forall u \neq v$, $\exists$ opens $U \ni u$, $V \ni v$ with $U \cap V = \emptyset$

The Zariski topology is not Hausdorff!

eg if X is irreducible, any two non empty open sets intersect.

Ex: A topological space X is Hausdorff if & only if the image of $\varphi: X \to X \times X$, $x \mapsto (x,x)$ is closed.

Defn Let $\varphi: X \to Y$ be a morphism of schemes.

The diagonal morphism

$\Delta: X \to X \times_Y X$ is the unique morphism induced by the identity morphisms $X \to X$.

$$
\begin{tikzcd}
X \arrow[r, "\exists! \Delta"] \arrow[rr, bend left] \arrow[dr] & X \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}
$$

Prop (Vakil 10.1.3)

Δ is a locally closed embedding (ie composition of a closed embedding & an open one)

Defn A morphism $\varphi: X \to Y$ is separated if the diagonal morphism is a closed embedding (equiv: if the image of Δ is a closed subset of $X \times_Y X$.)

A scheme X is separated if $\varphi: X \to \mathrm{Spec}(\mathbb{Z})$ is separated.

Separated $\Leftrightarrow$ image of Δ is closed

eg If $\varphi : X \to Y$ is a morphism of affine schemes then φ is separated

Write $X = \mathrm{Spec}(R)$, $Y = \mathrm{Spec}(S)$

$\Delta : X \to X \times_Y X$

comes from $\Delta : R \otimes_S R \to R$

$a \otimes b \mapsto ab$

This is surjective, so Δ is a closed embedding.

eg Let $X = \mathbb{A}^1_k$ "with the origin doubled".

This has an open cover $X = U \cup V$ with $U, V \simeq \mathbb{A}^1$, $U \cap V = \mathbb{A}^1 \setminus \{0\}$

X is not separated over $\mathrm{spec}(k)$

$X \times_k X$ is $\mathbb{A}^2$ with the axes doubled & 4 copies of the origin.

$\Delta : X \to X \times_k X$ has 2 of the origins in its image, but all 4 in the closure of the image.

Proper

Recall: In topology a continuous map $\varphi: X \to Y$ is proper if the preimage of a compact set is compact.

Defn A morphism is closed if the image of every closed set is closed.

A morphism $\varphi: X \to Y$ is universally closed if $\forall \psi: Y' \to Y$
$X \times_Y Y' \to Y'$ is closed
"remains closed after base change"

eg $\mathbb{A}^1_k \to \mathrm{Spec}(k)$ is closed

but $\underset{\mathbb{A}^2}{\mathbb{A}^1_k \times_k \mathbb{A}^1} \to \mathbb{A}^1_k$ is 

not closed

$$\begin{tikzcd} \mathbb{A}^2 \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d] \\ \mathbb{A}^1 \arrow[r] & \mathrm{Spec}(k) \end{tikzcd} \qquad (x,y) \mapsto x, \quad (x,y) \mapsto y$$

eg $V(xy-1)$ is closed in $\mathbb{A}^2$, but the image in $\mathbb{A}^1$ is $\mathbb{A}^1 \setminus \{0\}$.

Defn A morphism $\varphi: X \to Y$ is proper if it is separated, of finite type, universally closed.

If X "is a k-scheme" $X \to \mathrm{Spec}(k)$, then X is complete if it is proper over $\mathrm{Spec}(k)$.

If an author says "X is proper" they mean $\varphi: X \to Y$ is proper for an implied morphism (normally $Y = \mathrm{Spec}(k)$, sometimes $\mathrm{Spec}(\mathbb{Z})$)

Projective Schemes

Recall: a ring R is graded if $R \simeq \bigoplus_{i \geq 0} R_i$ with $R_i R_j \subseteq R_{i+j}$.

eg $R = K[x_0, x_n]$.

An ideal of I is homogeneous if it is generated by homogeneous elements (in some R_i).

Given a graded ring R, we construct a scheme $\mathrm{Proj}(R)$..

Set: Let $m = \bigoplus_{i > 0} R_i$

This is an ideal of R.

(eg $R = k[x_0, x_n]$, $m = \langle x_0, x_n \rangle$)

Points in $\mathrm{Proj}(R)$ are homogeneous ideals not containing m.

(eg $R = \mathbb{C}[x_0, x_1, x_2]$ $P = \langle 3x_0 - x_1, 5x_2 - x_2 \rangle$)

m is called the irrelevant ideal of R.

Zariski topology

Closed sets are $V(I)$ for I a homogeneous ideal.

$V(I) = \{P \mid I \subseteq P\}$ (I homog. $I \subseteq m$)

($I+J$, IJ are homogeneous if I, J are)

As for affine schemes, for a homogeneous $f \in R$,

$D(f) = \{P \in \mathrm{Proj}(R) \mid f \notin P\}$.

These form a basis for the topology on $\mathrm{Spec}(R)$.

Structure sheaf

Define $\mathcal{O}(D(f)) = (R_f)_0$

degree zero part.

Check that this is a sheaf on the base.

R_f inherits a grading from R:

$\deg\left(\frac{a}{f^m}\right) = \deg(a) - m\deg(f)$.

eg $R = k[x_0, \ldots, x_n]$ for k a field, $\deg(x_i) = 1 \ \forall i$

$m = \langle x_0, \ldots, x_n \rangle$.

$\mathrm{Proj}(R)$ consists of homog. primes not containing $m \Longleftrightarrow$ irreducible subvarieties of $\mathbb{P}^n$

For $f = x_i$

$(R_f)_0 = k[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$.

$\mathrm{Spec}((R_f)_0) = \mathbb{A}^n$.

For $f = x_0, \ldots, x_n$, this gives the standard affine cover of $\mathrm{Proj}(R) = \mathbb{P}^n$.

For a general projective scheme $\mathrm{Proj}(R)$,

$\mathrm{Proj}(R) = \bigcup D(f_i)$ $\quad m = \langle f_i \rangle$.

eg If I is a homogeneous ideal of R, then R/I is also graded.

(ie 2 homogeneous reps. of an element of R/I have the same degree

$\mathrm{Proj}(R/I)$ is a closed subscheme of $\mathrm{Proj}(R)$.