Last time:

Stalks of affine schemes

Locally ringed spaces
morphisms

Schemes
eg $\mathbb{A}^2 \setminus \{0\}$ is not an affine scheme

Defn A scheme $(X, \mathcal{O}_X)$ is connected if the the topological space X is connected

eg $R = \mathbb{C} \times \mathbb{C} \ni \{(a,b) \mid a,b \in \mathbb{C}\}$

$\mathrm{Spec}(R) = \{\mathbb{C} \times \{0\}, \{0\} \times \mathbb{C}\}$
with the discrete topology.

$R \simeq \mathbb{C}[x] / (x-1)(x-2)$

So $\mathrm{Spec}(R)$ is disconnected.

Defn A scheme $(X, \mathcal{O}_X)$ is irreducible if the topological space X is irreducible (cannot be written as a union of two proper closed subsets)

eg $R = \mathbb{C}[[x]]$ $\mathrm{Spec}(R)$
$= \{(0), \langle x \rangle\}$
$\langle x \rangle$ closed pt
$\overline{(0)} = \mathrm{Spec}(R)$.

$\mathrm{Spec}(R)$ is irreducible.

Defn A scheme is reduced if for every open set U the set $\mathcal{O}_X(U)$ has no nilpotent elements ($r \neq 0$ s.t. $r^n = 0$ for some n)

eg $R = \mathbb{C}[x]/x^2$

$\mathrm{Spec}(R)$ is not reduced.

eg $R = \mathbb{C}[x,y]/(xy)$

$\mathrm{Spec}(R) = \{ \langle x \rangle, \langle y \rangle,$

closed pts $\begin{cases} \langle x, y-a \rangle & a \in \mathbb{C} \\ \langle y, x-a \rangle & a \in \mathbb{C} \end{cases}$

$\mathrm{Spec}(R)$ is reduced.

<u>Ex</u> $\mathrm{Spec}(R)$ is reduced if & only if R has no nilpotents.

Warning: reduced & reducible are different concepts!

<u>Defn</u> A scheme X is integral if for every open set $U \subseteq X$ the ring $\mathcal{O}_X(U)$ is an integral domain (no zero divisors)

Prop (see II 3.1 Hartshorne)
A scheme is integral if and only if it is reduced and irreducible.

<u>Defn</u> A scheme X is locally Noetherian if it has an open affine cover $X = \bigcup \mathrm{Spec}(R_i)$ where each R_i is Noetherian. It is Noetherian if it is locally Noeth. & quasicompact. (so the cover above can be taken to be finite)

eg $X = \mathrm{Spec}(R)$, R Noetherian

non eg R valuation ring with non discrete valuation.

eg $\mathbb{C}\{\{t\}\}$ Puiseux series.

$3t^{-\frac{1}{2}} + 8t^{2} + 9t^{\frac{5}{3}} + \cdots$ (power series with complex coeff, fraction exponents with a common denom)

$\mathrm{val} : \mathbb{C}\{\{t\}\} \longrightarrow \mathbb{R}$

$\mathrm{val}(3t^{-\frac{1}{2}} + 8t^{2} + \cdots) = -\frac{1}{2}.$

$R = \{ f \mid \mathrm{val}(f) \geq 0 \}.$

R is not Noetherian, so $\mathrm{Spec}(R)$ is not

<u>Defn</u> A morphism $\varphi : X \to Y$ is locally of finite type if $\exists$ an open affine cover of Y $Y = \bigcup V_i$ with $V_i = \mathrm{Spec}(B_i)$ st

<u>Defn</u> A morphism
$\varphi : X \to Y$ is locally of finite type if $\exists$ an open affine cover of Y
$Y = \bigcup V_i$, $V_i = \mathrm{Spec}(B_i)$
s.t $\forall i$ $\varphi^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \mathrm{Spec}(A_{ij})$
s.t A_{ij} is a finitely generated B_i-algebra.

The morphism is of finite type if the cover $f^{-1}(V_i) = \bigcup U_{ij}$ can be chosen to be finite $\forall j$.

eg $R = \dfrac{\mathbb{C}[x,y]}{(x^2+y^2)}$

$\varphi : \mathrm{Spec}(R) \to \mathrm{Spec}(\mathbb{C})$
induced from
$\mathbb{C} \to \dfrac{\mathbb{C}[x,y]}{(x^2+y^2)}$

φ is of finite type.

in general
$\mathrm{Spec}(R) \to \mathrm{Spec}(\underset{\text{field}}{K})$
induced from $K \to R$ is of finite type if $R = \dfrac{K[x_1, \ldots x_n]}{I}$
for some n, I.

eg $f : X = \underbrace{\coprod_{i=0}^{\infty} \mathrm{Spec}(k)}_{\text{discrete topology}} \to \mathrm{Spec}(k)$

$f^{\#} : \mathcal{O}_Y \to \mathcal{O}_X$. diagonal map

$\mathcal{O}_X(U) = \prod_{p \in K} K$

<u>Defn</u> A morphism
$\varphi : X \to Y$ is finite
if there exists an open affine cover $Y = \bigcup V_i$, $V_i = \mathrm{Spec}(B_i)$ s.t
$\varphi^{-1}(V_i)$ is affine, $\varphi^{-1}(V_i) = \mathrm{Spec}(A_i)$
& A_i is a B_i-algebra that is f.g as a B_i-module.

eg $X = \mathrm{Spec}\left(\dfrac{K[x,y]}{y^2 - x^3 + x - 1} \right)$

$Y = \mathrm{Spec}(K[x])$.

$\varphi : X \to Y$. induced from
$K[x] \to \dfrac{K[x,y]}{(y^2 - x^3 + x - 1)}$ ← f.g $K[x]$ module. gen by $\{1, y\}$.

eg $X = \mathrm{Spec}(\mathbb{Q}(\sqrt{2}))$

$Y = \mathrm{Spec}(\mathbb{Q})$

$\varphi : X \longrightarrow Y$ from $\mathbb{Q} \hookrightarrow \mathbb{Q}(\sqrt{2})$.

$\mathbb{Q}(\sqrt{2})$ is a $\mathbb{Q}$-algebra that is generated by $\{1, \sqrt{2}\}$ as a $\mathbb{Q}$-module.

$\overset{\text{is}}{\mathbb{Q}[x]/(x^2-2)}$

<u>Intuition</u>: (locally) of finite type $\Rightarrow$ fibers are finite dimensional finite $\Rightarrow$ fibers are finite.

Defn Let K be an algebraically closed field.
A variety over k is an $\begin{pmatrix}\text{integral}\\ \text{reduced}\end{pmatrix}$ Noetherian scheme X of finite type over $\operatorname{Spec}(K)$. (ie $\exists$ morphism $X \to \operatorname{Spec}(K)$ of finite type)

Note: Some of the properties we've discussed (Noetherian, (locally) of finite type, finite) depend on a choice of of affine cover of Y, $f^{-1}(V)$.

Q, Really?

Lemma (Affine Communication Lemma)
Let P be a property of some affine open subsets of a scheme X s.t

i) if an affine open $\operatorname{Spec}(A) \to X$ has property P then $\forall f \in A$, $\operatorname{Spec}(A_f) \to X$ does as well.

ii) If $\langle f_1, f_n \rangle = A$ & $\operatorname{Spec}(A_{f_i}) \to X$ has P, then so does $\operatorname{Spec}(A) \hookrightarrow X$

Then If $X = \bigcup \operatorname{Spec}(A_i)$ where $\operatorname{Spec}(A_i)$ has property P $\forall i$, then every open affine does as well.

eg P on $\operatorname{Spec}(A)$ is "A is Noetherian".

i) If A is a Noetherian ring, A_f is as well.

ii) If $A = \langle f_1, , f_r \rangle$, A_{f_i} is Noetherian, then A is Noetherian.

(Idea: If I is an ideal of A, IA_{f_i} is fg $\forall i$. Take generators $\{a_{ij}\}$ for all of these.
Set $\overset{I}{J} = \langle a_{ij} \rangle \leq A$.
Consider $a \in I \setminus J$ in A/J.
$\frac{a}{1} = 0$ in $(A/J)_{f_i}$ $\forall i$, so $a = 0$

Lemma If $\mathrm{Spec}(A)$ & $\mathrm{Spec}(B)$ are open affine subschemes of a scheme X, then $\mathrm{Spec}(A) \cap \mathrm{Spec}(B)$ is the union of open sets that are simultaneously basic opens for $\mathrm{Spec}(A)$ & $\mathrm{Spec}(B)$.

$f \in A$
$g \in B$

Pf Fix a pt $p \in \mathrm{Spec}(A) \cap \mathrm{Spec}(B)$ & $f \in A$ with $p \in \mathrm{Spec}(A_f)$

There is then $g \in B$ with $p \in \mathrm{Spec}(B_g) \subseteq \mathrm{Spec}(A_f)$.

Let $g' = \underset{\mathcal{O}_X(\mathrm{Spec}(B)) \,=\, B}{g}\Big|_{\mathrm{Spec}(A_f)} \in \mathcal{O}_X(\mathrm{Spec}(A_f)) \cong A_f$

$$\mathrm{Spec}(B_g) \underset{\text{key pt}}{=} \mathrm{Spec}(A_f) \setminus \{p \mid g' \in p\}$$
$$= \mathrm{Spec}((A_f)_{g'})$$
$$= \mathrm{Spec}(A_{fg''}) \quad g' = \frac{g''}{f^m}$$

Pf of the ACL

Let $\mathrm{Spec}(A)$ be an affine open subscheme of X. Cover $\mathrm{Spec}(A)$ with basic opens $\mathrm{Spec}(A_{g_i})$ each of which is in some $\mathrm{Spec}(A_j)$. (We may assume $\mathrm{Spec}(A_{g_i})$ is a basic open of $\mathrm{Spec}(A_j)$)

We may also assume the cover is finite (affine schemes are quasicompact)

By i) Each $\mathrm{Spec}(A_{g_i})$ has P.

By ii) $\mathrm{Spec}(A)$ has P.

<u>Defn</u> A morphism $\varphi: X \to Y$ is affine if there is an open cover $Y = \bigcup \mathrm{Spec}(A_i)$ s.t. $\varphi^{-1}(\mathrm{Spec}(A_i))$ is affine $\forall i$.

(Harder ex: This satisfies the conditions of the ACL)

<u>Recall</u>: An open subscheme of X is a scheme isomorphic to $(U, \mathcal{O}_X|_U)$ for an open $U \subseteq X$.

<u>Defn</u> An open embedding (immersion) is $f: X \to Y$ that induces an isomorphism of X with an open subscheme of Y.

<u>Note</u>: There's a natural scheme structure on an open subset of X, but not on a closed set.

eg $X = \mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[x,y])$

$V = x\text{-axis} = V(y)$
$= \{\langle y\rangle, \langle y, x-a\rangle\}$

As a set, this $= \mathrm{Spec}\left(\mathbb{C}[x,y]/(y^m)\right)$ for any m.

eg $X = \mathrm{Spec}\left(\mathbb{C}[x]/(x^5)\right)$

$\overset{\text{as set only}}{=} \mathrm{Spec}\left(\mathbb{C}[x]/(x^i)\right)$, $1 \leq i \leq 5$.

<u>Defn</u> A morphism $\pi: X \to Y$ is a closed embedding if it is an affine morphism & for every $\mathrm{Spec}(B) \subseteq Y$, $\pi^{-1}(\mathrm{Spec}(B)) = \mathrm{Spec}(A)$ the map $B \to A$ is surjective ie $A = B/I$ for some ideal I of B.