Last time:

- Sheafification.
- Kernels/images/cokernels of morphisms of sheafs
- Affine schemes.

Given R, $\mathrm{Spec}(R) = \{\text{primes in } R\}$

Zariski topology:

closed sets $V(I) = \{P \supseteq I\}$

$\{D(f) = \{P \mid f \notin P\}\}$ is a basis for the topology

Sheaf on this basis

$$\mathcal{O}_{\mathrm{Spec}(R)}(D(f)) = R_f$$

Prop a) $(\mathcal{O}_{\mathrm{Spec}(R)})_p = R_p$

localization of R at p.

b). $\mathcal{O}_{\mathrm{Spec}(R)}(U)$

$= \{$fcns $s : U \longrightarrow \coprod_{p \in U} R_p$

s.t $s(P) \in R_p$,

$\forall p \in U \ \exists V$,

$p \in V \subseteq U$

and $r, f \in R$ s.t

$\forall Q \in V$, $f \notin Q$, and

$s(Q) = \frac{r}{f} \in R_Q \}$

a) The stalk $\mathcal{F}_p$ of a sheaf $\mathcal{F}$ coming from a sheaf on a basis is

$$\{ (s, B \mid s \in \mathcal{F}(B), p \in B \} / \sim$$

(given (s, U), $\exists\, B \in \mathcal{B}$ s.t $p \in B \subseteq U$. $(s, U) \sim (s|_B, B)$)

For $P \in Spec(R)$, we have $P \in D(f)$ if & only if $f \notin P$.

Construct

$$\varphi : R_P \longrightarrow (\mathcal{O}_{Spec(R)})_P$$

by $\varphi\left(\frac{r}{f}\right) = \left(\frac{r}{f}, D(f)\right)$

Check:

1) This is well-defined

2) This is a ring homomorphism.

$$\varphi\left(\frac{r_1}{g_1}\frac{r_3}{g_2}\right) = \left(\frac{r_1 r_2}{g_1 g_2}, D(g_1 g_2)\right)$$

$$= \left(\frac{r_1 g_2}{g_1 g_2}, D(g_1 g_2)\right) \cdot \left(\frac{r_2 g_1}{g_1 g_2}, D(g_1 g_2)\right)$$

$$= \left(\frac{r_1}{g_1}, D(g_1)\right) \cdot \left(\frac{r_3}{g_2}, D(g_2)\right)$$

3) φ is surjective:

if $(s, D(f)) \in (\mathcal{O}_{Spec(R)})_P$

then $s = \frac{r}{f^m}$ for some $r \in R$, $m \geq 0$, $f \notin P$, so $\frac{r}{f^m} \in R_P$, & $\varphi\left(\frac{r}{f^m}\right) = \left(\frac{r}{f^m}, D(f^m)\right) = (s, D(f))$

4) φ is injective:

if $\varphi\left(\frac{r}{g}\right) = \varphi\left(\frac{r'}{g'}\right)$ then

$$\left(\frac{r}{g}, D(g)\right) \sim \left(\frac{r'}{g'}, D(g')\right)$$

So $\exists\, h \in R$, $h \notin P$, with $p \in D(h) \subseteq D(g) \cap D(g')$ s.t $\frac{r}{g}\Big|_{D(h)} = \frac{r'}{g'}\Big|_{D(h)}$.

We have $h^m = sg$, $h^m = s'g'$ for $s, s' \notin P$

so $\frac{rs}{h^m} = \frac{r's'}{h^m} \in R_h$

Thus $h^N(rs - r's') = 0 \in R$ for some N.

So (multiply by h^m)

$$h^N(rss'g' - r's'sg) = 0$$

$$= \underbrace{h^N ss'}_{\notin P}(rg' - r'g) = 0 \text{ in } R$$

so $\frac{r}{g} = \frac{r'}{g'} \in R_P$.

b) This is just the defn of recovering a sheaf from a sheaf on the basis $\{D(f)\}$

<u>Note</u>: This means that the stalk of $\mathrm{Spec}(R)$ is a <u>local</u> ring

<u>Defn</u> Let $f: X \to Y$ be a continuous map of top. spaces, & let $\mathcal{F}$ be a sheaf on X. The direct image sheaf $f_*\mathcal{F}$ on Y is given by

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$$

Morphisms

<u>Defn</u> A ringed space is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X & a sheaf of rings $\mathcal{O}_X$ on X.

A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ of a cts map $f: X \to Y$ & a morphism

$$f^\#: \mathcal{O}_Y \to f_*\mathcal{O}_X$$

of sheaves of rings on Y.

eg If $\varphi: R \to S$ is a ring homomorphism then we get an induced morphism

$$(\mathrm{Spec}(S), \mathcal{O}_{\mathrm{Spec}(S)}) \to (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$$

$$\varphi: \mathrm{Spec}(S) \to \mathrm{Spec}(R)$$
$$P \mapsto \varphi^{-1}(P)$$

check that this prime.

φ is continuous:

If $V(I)$ is closed in Spec(R), then

ring hom.

$$\varphi^{-1}(V(I)) = \{P \in \mathrm{Spec}(S) \mid \varphi^{-1}(P) \in V(I)\}$$
$$= \{P \in \mathrm{Spec}(S) \mid I \subseteq \varphi^{-1}(P)\}$$
$$= \{P \in \mathrm{Spec}(S) \mid \varphi(I) \subseteq P\}$$
$$= V(\varphi(I))$$

so preimages of closed sets are closed.

Define $\varphi^\#: \mathcal{O}_{\mathrm{Spec}(R)} \to \varphi_*\mathcal{O}_{\mathrm{Spec}(S)}$ on $D(f)$ for $f \in R$ by

$$\underbrace{\varphi^\#(D(f))}_{D(f) \to D(\varphi(f))}\left(\frac{a}{f}\right) = \underbrace{\frac{\varphi(a)}{\varphi(f)} \in S_{\varphi(f)}}_{\varphi^{-1}(D(f)) = D(\varphi(f))}.$$

Defn A ringed space $(X, \mathcal{O}_X)$ is a locally ringed space if for every point $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a local ring.

(eg $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$)

A morphism $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ induces a ring homorphism

$f_p^\# : \mathcal{O}_{Y, f(p)} \longrightarrow \mathcal{O}_{X,p}$ by

$$(s, U) \longmapsto (f^\#(U)(s), f^{-1}(U))$$

$U \ni f(p)$, $p \in f^{-1}(U)$

$s \in \mathcal{F}(U)$ $\quad f^\#(U) : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$

(check well-defined, ring homomorphism)

A morphism of locally ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism $(f, f^\#)$ of ringed spaces st $\forall p$, $f_p^\#$ is a local homomorphism of local rings.

always have $\subseteq$

$f_p^{\#-1}(\text{max ideal}) = \text{max ideal}$.

eg $(\varphi, \varphi^\#) : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$. is a morphism of locally ringes spaces.

& $\varphi_p^\# : R_{\varphi^{-1}(p)} \longrightarrow S_p$ is given by $\frac{a}{f} \mapsto \frac{\varphi(a)}{\varphi(f)}$.

In fact

morphisms of locally ringed spaces $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$

$\updownarrow$

ring homomorphisms $R \to S$

(needs locally ringed)

(see Hartshorne prop 2.3c)

<u>Defn</u> An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic (as a locally ringed space) to $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ for some R.

A scheme is a locally ringed space $(X, \mathcal{O}_X)$ for which every point p has an open nbhd U st $(U, \mathcal{O}_X|_U)$ is an affine scheme.

We call X the underlying space and $\mathcal{O}_X$ the structure sheaf.

A morphism of schemes is a morphism of locally ringed spaces. An isomorphism is a morphism with a two-sided inverse.

eg $\mathbb{A}^2 \setminus \{0\}$. Fix a field k. Let $R = k[x, y]$ & $X = \mathrm{Spec}(R)$.

Let $U = D(x) \cup D(y)$
$= \underbrace{\mathrm{Spec}(R)}_{\mathbb{A}^2} \setminus \{\langle x, y\rangle\}$.

Consider $(U, \mathcal{O}_X|_U)$

This is a scheme:
If P is a pt of U, then wlog $P \in D(x)$, and the nbhd $(D(x), \mathcal{O}_X|_{D(x)})$
$\simeq (\mathrm{Spec}(R_x), \mathcal{O}_{\mathrm{Spec}(R_x)})$

<u>Note:</u> U is not affine.
To see this, consider $\mathcal{O}_U(U)$. We claim this is $R = k[x, y]$.

By definition it is

$$\left\{ s : U \longrightarrow \coprod_{p \in U} R_p \;\middle|\; \begin{array}{l} s(p) \in R_p, \\ \forall p \in U \\ \exists p \in D(f) \subseteq U \\ \& r \in R, m > 0 \text{ st} \\ \forall Q \in D(f), s(Q) = \frac{r}{f^m} \in R_Q \end{array} \right\}$$

For $s \in \mathcal{O}_U(U)$ we have
$s|_{D(x)} = \frac{r}{x^n} \in R_x$ and $\frac{r}{x^n}\Big|_{D(xy)} = \frac{r'}{y^n}\Big|_{D(xy)}$
$s|_{D(y)} = \frac{r'}{y^n} \in R_y$

ie (wlog $m>n$)

$$\frac{ry^m}{x^m y^n} = \frac{r' y^{n-m} x^m}{x^m y^n}$$

So $ry^m = r' y^{n-m} x^m$

Since R is a UFD,

$x^m | r$ $\quad r = x^m \tilde{r}$.

$\tilde{r} y^m = r' y^{n-m}$

So $\frac{r}{x^m} = \frac{\tilde{r}}{1} = \frac{r'}{y^n}$ and

thus $s(P) = \frac{\tilde{r}}{1} \in R_P$. $\forall P \in U$

Since any $r \in R$ gives an element of $\mathcal{O}_U(U)$ by restriction we have $\mathcal{O}_U(U) \simeq R = k[x,y]$

This shows that U is not affine.

If $(U, \mathcal{O}_U) \simeq (\mathrm{Spec}(S), \mathcal{O}_{\mathrm{Spec}(S)})$ for some ring S, then $S = \mathcal{O}_{\mathrm{Spec}(S)}(\mathrm{Spec}(S)) \simeq \mathcal{O}_U(U) = R$

So there must be

$(\psi, \psi^\#) : (U, \mathcal{O}_U) \xrightarrow{\simeq} (\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$

ψ takes $U \setminus D(x)$, $U \setminus D(y)$ to closed sets in $\mathrm{Spec}(R)$ that do not intersect.

ie $V(I)$, $V(J)$ with $V(I+J) = \phi$.

$I + J = R$, $\exists r \in I$, $1 - r \in J$.

This contradicts $\langle x \rangle + \langle y \rangle \neq R$.

since we can recover I from $V(I)$ $I = \bigcap_{P \in V(I)} P$,

& $\psi^\#(\mathrm{Spec}(R))(I) = \langle x \rangle$

(since we can recover $\langle x \rangle$ from $U \setminus D(x)$)