Lecture 2

Recap

Last time we talked about presheaves and sheaves.

Presheaf: $\mathcal{F}$ on a topological space X, with groups (resp. rings, sets, etc.) $\mathcal{F}(U)$ for each open set $U \subset X$, with restriction homs $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$ for all open $V \subset U$, satisfying certain conditions ($\rho_{UU} = \mathrm{id}_U, \rho_{UW} = \rho_{VW}\rho_{UV}, \mathcal{F}(\varnothing) = 0$).

Sheaf: In addition, we also require identity and gluing.

Identity: Given $U = \cup U_i, s \in \mathcal{F}(U)$ s.t. $s|_{U_i} = 0 \ \forall i$, then $s = 0$. (locally zero everywhere $\Rightarrow$ zero)

Gluing: Given $s_i \in \mathcal{F}(U_i)$ s.t. $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \ \forall i, j$, then $\exists s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$. (suffices to give a function locally, agreeing on overlaps)

Remark: $\mathcal{F}(\varnothing) = 0$ is not implied by either the presheaf conditions, nor by considering presheaves as a contravariant functor Op(X)$\to$Ab. It is applied axiomatically by convention. For sheaves, it is implied by gluing (e.g. consider a cover of the empty set). Moral: don't worry about it.

Morphisms of (Pre)sheaves: $\phi : \mathcal{F} \to \mathcal{G}$. A series of morphisms $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for every open set U, that commute with restriction maps:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\phi(U)} & \mathcal{G}(U) \\ \rho\big\downarrow & \circlearrowleft & \big\downarrow\rho \\ \mathcal{F}(V) & \xrightarrow[\phi(V)]{} & \mathcal{G}(V) \end{array}$$

A morphism is an isomorphism if it has an inverse.

Stalks: $\mathcal{F}_p = \varinjlim_{p \in U} \mathcal{F}(U) = \coprod \mathcal{F}(U)/\sim \, = \{(s, U) | s \in \mathcal{F}(U)\}/\sim$, where $(s, U) \sim (s', U')$ if $\exists V \subset U \cup U'$ such that $s|_V = s'|_V$.

More sheaves

Stalks package most of the information we want. A morphism of (pre)sheaves $\phi : \mathcal{F} \to \mathcal{G}$ induces a homomorphism on stalks, $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$, i.e. $\phi_p(s, U) = (\phi(U)s, U)$. This is well-defined, for if $(s, U) = (s', U')$, then because ϕ commutes with restrictions, $\phi_p(s, U) = \phi_p(s', U')$. This gives another characterization of isomorphisms.

Proposition. *(H, Prop.II.1.1): Let $\mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then ϕ is an isomorphism if and only if ϕ_p is an isomorphism $\forall p \in X$.*

Sheafification: Every presheaf has an associated sheaf. Given a presheaf $\mathcal{F}$, we define a sheaf $\mathcal{F}^+$ called the *sheafification* of $\mathcal{F}$, as follows:

$$\mathcal{F}^+(U) = \{\text{functions } s : U \to \coprod \mathcal{F}_p \,|\, s(p) \in \mathcal{F}_p, \text{ and } \forall p \in U, \exists V \text{ with } p \in V \subset U \text{ and } t \in \mathcal{F}(V)$$
$$\text{such that } s(q) = t_q \, \forall q \in V, \text{ where } t_q = (t, V) \in \mathcal{F}_q\}$$

These are functions which map into the disjoint union of the stalks, where we require that p is mapped into the stalk at p, and locally the choices of stalks are related. Usually presheaves fail to be sheaves when they don't satisfy the gluing axiom. The + construction fixes this by adding in enough elements to allow gluing.

Exercise:

1. $\mathcal{F}^+$ is a sheaf.
2. There is a natural morphism of presheaves $\theta : \mathcal{F} \to \mathcal{F}^+$ such that $s \in \mathcal{F}(U) \mapsto s^+ : U \to \coprod \mathcal{F}_p$, with $s^+(p) = s_p = (s, U)$.
3. Universal property. For any sheaf $\mathcal{F}$ and a morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ there exists a unique $\psi : \mathcal{F}^+ \to \mathcal{G}$ such that:

$$\begin{array}{ccc} \mathcal{F} & \longrightarrow & \mathcal{F}^+ \\ & \searrow & \downarrow \exists! \psi \\ & & \mathcal{G} \end{array}$$

$\mathcal{F}^+$ is the simplest possible sheaf we could associate to $\mathcal{F}$. The + construction is useful because sometimes when we do an operation on a sheaf, we only get a presheaf, so then we need to sheafify.

Definition: Let $\phi : \mathcal{F} \to \mathcal{G}$ be a morphism of (pre)sheaves.
The presheaf kernel of ϕ is $U \mapsto \ker \phi(U)$
The presheaf cokernel of ϕ is $U \mapsto \text{coker } \phi(U) = \mathcal{G}(U)/\phi(U)$
The presheaf image of ϕ is $U \mapsto \phi(U)$

Each of the $\phi(U)$ are group homomorphisms, so it makes sense to talk about kernels, etc. These are subpresheaves of $\mathcal{F}$ (for $\ker \phi$) or $\mathcal{G}$ (for the other two).

If $\mathcal{F}$ and $\mathcal{G}$ are sheaves, then presheaf kernel is in fact a sheaf (check this!), and the sheaf *kernel* ($\ker \phi$) is defined as this. But the sheaf *cokernel* (coker ϕ) and sheaf *image* (im ϕ) are the sheafifications of the presheaf cokernel and images. Surjectivity doesn't mean what you think it does.

We say $\phi : \mathcal{F} \to \mathcal{G}$ is *injective* if $\ker \phi = 0$.
We say $\phi : \mathcal{F} \to \mathcal{G}$ is *surjective* if im $\phi = \mathcal{G}$.

Affine Schemes

From now on, R will always be a commutative ring with identity (not necessarily Noetherian).

To talk about geometry, we need a topological space and a notion of functions. Sheaves give us a way of talking about functions.

Definition: An *affine scheme*, Spec R, is a topological space, with a sheaf of rings $\mathcal{O}_{\text{Spec}R}$, its *structure sheaf.*

Examples to keep in mind when thinking about these definitions: i) think of R as a coordinate ring of a variety - or even $k[x_1, ..., x_n]$; ii) think of R as the most pathological example imaginable.

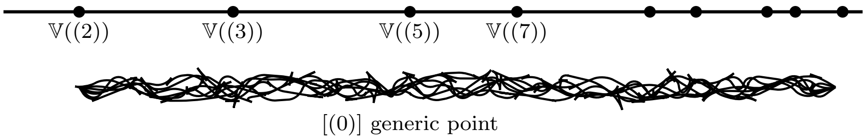
A *sheaf of rings* means a sheaf where $\mathcal{O}_{\text{Spec}R}(U)$ is a ring $\forall$ open U and the homs are ring homs.

The topological space Spec R is the set of all prime ideals in R. We place the Zariski topology on Spec R : the closed sets are $V(I) = \{P : P \geq I\}$ for an ideal $I \leq R$.

(Check!) This is a topology. $V(I \cap J) = V(I) \cup V(J)$ and $V(I + J) = V(I) \cap V(J)$. Proof is the same as for varieties.

Examples:

1. $R = \mathbb{Z}$. Spec $\mathbb{Z} = \{0\} \cup p\mathbb{Z}$, p prime.



   Notice we have closed points (e.g. 2, 3, 5,...) and not closed points (0, whose closure is Spec $\mathbb{Z}$ because $0 \in (p), \forall (p)$).

2. $R = k[x]$. The prime ideals are (f), f irreducible. If $k = \overline{k}$, then Spec $R = (0), (x-a), a \in k$. $(x-a)$ are the closed points; naively they look like $\mathbb{A}^1$. We also have the generic point (0). If $k = \mathbb{R}$, then Spec $R = (0), (x-a)$, and $(x^2 + ax + b)$ such that $a^2 - 4b < 0$. Again the $(x-a)$ and $(x^2 + ax + b)$ are closed points, and the closure of (0) is Spec R. If $k = \mathbb{Z}/2\mathbb{Z}$, then Spec R is infinite $= (0), (x), (x+1), (x^2+x+1), ...$ Again (0) is a generic point, the others are closed. Observe that $\mathbb{A}^1$ here is not a two point set, as might be expected. The space remembers the orbits of points over the algebraic closure.

3. $R = k[x_1, ..., x_n], k = \overline{k}$. If (0) is prime (integral domain) it will always be dense. $(x_1 - a_1, ..., x_n - a_n)$ are closed points (0satz). Any irreducible subvariety of $\mathbb{A}^n$ corresponds to a prime ideal. The closure of a such a prime contains all the points on it.

The sheaf of rings. To give the sheaf of rings on our topological space, we make use of a basis for the topology. For any $f \in R$, we have the basic open set $D(f) = \{P \in \text{Spec}R : f \notin P\}$. The set of all $D(f)$ forms a basis for the Zariski topology. (cf. HW1 Q4) Given a sheaf we can recover it by just knowing it on a basis. We'll define the structure sheaf $\mathcal{O}_{\text{Spec}R}$ by defining it on the basis, i.e. we'll give a ring for every $D(f)$ such that it gives a sheaf on the base (check restrictions, id, gluing).

Define: $\mathcal{O}_{\text{Spec}R}(D(f)) = R_f$.

R_f is R localized at the multiplicatively closed set (a set closed under products, including the empty product 1) $\{1, f, f^2, ...\}$. That is, $R_f = \{\frac{r}{f^m} | r \in R, m \geq 0\}/\sim$, where $\frac{r}{f^m} \sim \frac{r'}{f^n}$ if $\exists j$ such that $f^j(f^n r - f^m r') = 0$. It's like fractions, but a more general equivalence relation to take care of zero divisors.

In particular, if $f = 1$, $\mathcal{O}_{\text{Spec}R}(D(1)) = R$, called the global sections. In the case R is the coordinate ring of a variety, the polynomial functions on a variety, then in a naive variety sense, $D(f)$ is the set of points where f is nonzero, so rational functions with f in the denominator are still well-defined functions on the variety.

Now we want to check this definition actually gives us a sheaf on the base. For this we start with a couple of commutative algebra results.

Lemma. *If I is an ideal of R disjoint from a multiplicatively closed set U, then an ideal P, maximal with respect to containing I and disjoint from U, is prime.*

Proof. If $f, g \notin P$, then $P + (f), P + (g)$ are larger ideals containing I, so $\exists u_1 \in P + (f), u_2 \in P + (g)$, with $u_1, u_2 \in U$. So, $u_1 = p_1 + r_1 f, u_2 = p_2 + r_2 g$. Since $u_1 u_2 \notin P$, then $fg \notin P$. □

Corollary. *If $D(f) \subseteq D(g)$, then $\exists m > 0$ such that $f^m \in (g)$.*

Proof. If not, (g) is disjoint from $U = \{1, f, f^2, ...\}$ so by the lemma $\exists P \in D(f) \setminus D(g)$. □

Thus, if $D(f) \subseteq D(g)$, we have $f^m = gr$ for some r, so we have a homomorphism

$$R_g \to R_f, \qquad \frac{a}{g^n} \mapsto \frac{ar^n}{g^n r^n} = \frac{ar^n}{f^{mn}}$$

so restriction is straightforward. When $D(f) \subseteq D(g) \subseteq D(h)$, this satisfies

$$\begin{array}{ccc} R_h & \longrightarrow & R_f \\ & \searrow \quad \nearrow & \\ & R_g & \end{array}$$

This is a 'presheaf on the base'. We now check this presheaf is a sheaf on the base, that it satifies identity and gluing.

Suppose now that we have an open cover $D(f) = \bigcup_{i \in I} D(f_i)$.

First, we claim that $\exists m > 0$ such that $f^m \in (f_i, i \in I)$. Otherwise, the lemma gives $P \geq (f_i)$ avoiding $U = \{1, f, f^2, ...\}$, which is a contradiction. Also, as before, since $D(f_i) \subseteq D(f)$, we can write $f_i^{m_i} = fr_i$ for some m_i, r_i $\forall i \in I$.

Identity: (section vanishes if all its restrictions do). Suppose $a/f^n \in R_f$ satisfies $ar_i^n/f_i^{m_i n} = \frac{0}{1} \in R_{f_i}$ $\forall i$. Then $\exists N_i$ such that $f_i^{N_i} a r_i^n = 0$. So $0 = f^n f_i^{N_i} a r_i^n = f_i^{N_i + m_i n} a$.

Since $f^m \in (f_i, i \in I)$, then for some $N \gg 0$, $f^N \in (f_i^{N_i + m_i n}, i \in I)$. So $f^N = \sum b_i f_i^{N_i + m_i n}$ for some $b_i \in R$.

Then $f^N a = \sum b_i f_i^{N_i + m_i n} a = 0$ and $a/f^n = 0/1$ in R_f. □

Gluing: (Given sections of $D(f_i)$ agreeing on overlaps, can construct a section of $D(f)$). Notice that our open cover $D(f) = \bigcup_{i \in I} D(f_i)$ may be assumed finite , although we haven't made any assumptions about Noetherianness of rings. Elements of ideals are only finite sums, $f^m = \sum_{i \in J} b_i f_i, |J| < \infty$. This implies $D(f) = \bigcup_{i \in J} D(f_i)$, because any prime not containing f^m, and therefore f, must omit one of the $f_i, i \in J$. Every cover does have a finite subcover ('quasi-compactness'); but we will see later that properness is a more useful generalization of compactness.

Claim: It suffices to check gluing only on finite covers. Suppose we are given $s_i = \frac{a_i}{f_i^{n_i}} \in R_{f_i}, \forall i \in I$ with $\frac{a_i f_j^{n_i}}{(f_i f_j)^{n_i}} = \frac{a_j f_i^{n_j}}{(f_i f_j)^{n_j}} \in R_{f_i f_j}$. It then suffices to show there is $s \in R_f$ with $s|_{D(f_i)} = s_i, \forall i \in J$. Identity will imply $s|_{D(f_i)} = s_i, \forall i \in I$ (think about this). □(claim)

The finite cover enables us to take a maximum over the exponents that show up.

Since $s_i|_{D(f_i) \cap D(f_j)} = s_j|_{D(f_i) \cap D(f_j)}, \forall i, j \in J, \exists \tilde{N}$ such that $(f_i f_j)^{\tilde{N}} (f_j^{n_j} a_i - f_i^{n_i} a_j) = 0$ in R. We use finiteness here - for pairs, we can make fractions have the same denominator, so assume all pairs have the same denominator. Then since J is finite, renaming a_i's if necessary, we may assume that $n_i = n_j = N, \forall i, j$ and $\tilde{N} = 0$, i.e. $f_j^N a_i - f_i^N a_j = 0$ in R.

We can write $f^m = \sum_{i \in J} c_i f_i^N$ for some $m \gg 0$, and set $g = \sum_{i \in J} c_i a_i$.

We claim $\frac{g}{f^m}|_{D(f_i)} = s_i$.

Indeed $f_i^N g = \sum_{j \in J} f_i^N c_j a_j = \sum_{j \in J} c_j a_i f_j^N = a_i f^m$, and so (recall $f_i^{m_i} = fr_i$)

$\frac{g}{f^m}|_{D(f_i)} = \frac{g r_i^m}{f_i^{m_i m}}|_{D(f_i)} = \frac{a_i}{f_i^N} = s_i$. So we have gluing. □

We have now shown that we do have a sheaf on the base.

We could also have done this another way. The approach in H is to give the following definition, and this is then proved to be the same as the way we defined it using a base. We will discuss this further next time. Also, recall that the localization at a prime ideal R_p is defined with $R \setminus p$ as the multiplicatively closed set .

Proposition.

a) The stalk $(\mathcal{O}_{SpecR})_p$ *of* $(\mathcal{O}_{SpecR})$ *at* p *is the localization* R_p.

b) $(\mathcal{O}_{SpecR})(U)$ *is the functions*

$$\{s : U \to \coprod R_p |\, s(p) \in R_p;\, and\, \forall p \in U, \exists V \subseteq U\, and\; a, f \in R,$$
$$such\; that\; \forall Q \in V\, and\; f \notin Q, s(Q) = a/f \in R_q\}$$