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Module webpage exists

Recall: $\mathbb{A}^n = \mathbb{A}^n_K = K^n$

K a field.

Let I be an ideal in $K[x_1, \dots, x_n]$, the variety of I is

$$V(I) = \{y \in \mathbb{A}^n \mid f(y) = 0 \ \forall f \in I\}$$

The ideal of $X = V(I)$ is

$$I(X) = \{f \in K[x_1, \dots, x_n] \mid f(y) = 0 \ \forall y \in X\}$$

<u>Nullstellensatz</u>:

$K = \overline{K}$, then $I(V(I)) = \sqrt{I}$

$$\sqrt{I} = \{f \mid \exists m > 0 \text{ with } f^m \in I\}$$

Coordinate ring: $k[X]$

$= k[x_1, \ldots, x_n] / I(X)$

Affine varieties $/ K$

$\Updownarrow$

f.g. K-algebras with no nilpotents.

First description:

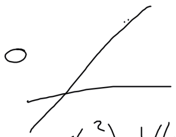
1) What if we replace $I(X)$ by a general ideal J?

2) Replace K by a general ring?

3) Go beyond quasiprojective varieties.

eg $V(y(y-ax)) \subseteq \mathbb{A}^2$

$a \in K$.

$a \neq 0$  two lines

one line.

$a = 0$ $V(y^2) = V(y)$

eg $K = \mathbb{Q}$

$V(x^3 + y^3 - 1) \subseteq \mathbb{A}^2_{\mathbb{Q}}$

Second description.

A geometric space is a set with functions on it.

eg $X = V(I)$ fcns $k[X]$

eg X smooth manifold fcns are differentiable fcns.

eg X top. manifold fcns cts fcns

Schemes are a choice of space/fcns that works well for algebraic geom

Refs: Hartshorne

Eisenbud-Harris
"The geometry of schemes"

Vakil's notes
Ravi Vakil @ stanford

Credit:
Weekly homework!

Do on average 3 problems.

Email me today!

Sheaves

Let X be a topological space & for all open $U \subseteq X$ let $\mathcal{O}(U)$ be the set of continuous functions $f: U \to \mathbb{R}$.

Notes: 1) We have restriction maps $res_{UV}: \mathcal{O}(U) \to \mathcal{O}(V)$ if $V \subseteq U$ is open

2) If $W \subseteq V \subseteq U$ is open

$$\begin{tikzcd} \mathcal{O}(U) \arrow[rr, "res_{UW}"] \arrow[dr, "res_{UV}"'] & & \mathcal{O}(W) \\ & \mathcal{O}(V) \arrow[ur, "res_{VW}"'] & \end{tikzcd}$$

commutes

3) If $U = \bigcup U_i$ is an open cover of U & $f, g \in \mathcal{O}(U)$ with $res_{U,U_i}(f) = res_{U,U_i}(g)$ $\forall i$, then $f = g$.

Equiv: if $res_{U,U_i}(f) = 0$ $\forall i$

4) If $U = \bigcup U_i$ is an open cover of U, and $f_i \in \mathcal{O}(U_i)$ is given

4) If $U = \bigcup U_i$ is an open cover of U, & $f_i \in \mathcal{O}(U_i)$ is given with $res_{U_i, U_i \cap U_j}(f_i) = res_{U_j, U_i \cap U_j}(f_j)$

"agrees on overlaps"

Then $\exists f \in \mathcal{O}(U)$ with $res_{U,U_i}(f) = f_i$

Defn Let X be a topological space. A presheaf $\mathcal{F}$ of abelian groups on X consists of:

a). For every open set $U \subseteq X$, an abelian group $\mathcal{F}(U)$

b) For every inclusion $V \subseteq U$ of open sets a homomorphism $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$

s.t. 0) $\mathcal{F}(\emptyset) = 0$

1) ρ_{UU} is the identity map

2) if $W \subseteq V \subseteq U$, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$

We call an element of $\mathcal{F}(U)$ a section of $\mathcal{F}$ on U.

& $\rho_{UV}(s)$ is the restriction of s to V – sometimes write $s|_V$

Categorical interpretation

Let $\mathcal{T}op(X)$ be the category with objects open sets of X & morphisms inclusions (ie $\mathrm{Hom}(U,V)$ is a pt if $U \subseteq V$ & $\emptyset$ o/w.

A presheaf is a contravariant functor $\mathcal{T}op(X) \longrightarrow$ Abelian groups.

(References: Ch 1 Vakil
Leinster (CUP)

eg $X = \{x, y\}$ with the discrete topology

$$\begin{tikzcd} & \{x\} \; {\scriptstyle G/G_1,\, G_3} \arrow[dr] \arrow[dl, shift left] & \\ \emptyset \; {\scriptstyle 0} \arrow[ur] \arrow[dr] & & X \; {\scriptstyle G} \arrow[ul, shift left] \arrow[dl, shift left] \\ & \{y\} \; {\scriptstyle G/G_2,\, G_2} \arrow[ur] \arrow[ul, shift left] & \end{tikzcd}$$

A presheaf on X is given by an abelian group G_1 and two subgroups (or inclusion $G/G_1 \to G_3$

eg A any abelian group
$\mathcal{F}(U) = \begin{cases} A & U \neq \emptyset \\ 0 & U = \emptyset \end{cases}$ is the constant presheaf

<u>Defn</u> A presheaf $\mathcal{F}$ on a topological space X is a <u>sheaf</u> if it satisfies

3) If U is an open set & $\bigcup U_i$ is an open cover & $s \in \mathcal{F}(U)$ satisfies $\underset{\rho_{UU_i}(s)}{s|_{U_i}} = 0 \ \forall i, \dots$ then $s = 0$

4) If U is an open set & $\{U_i\}$ is an open covering of U & $s_i \in \mathcal{F}(U_i)$ satisfies
$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \quad \forall i, j$$
then $\exists s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$
((3) $\Rightarrow$ s is unique).

Not every presheaf is a sheaf.

eg constant presheaf on 2pt space

$$\begin{tikzcd} & \{x\}\ \mathbb{Z}/2\mathbb{Z} \arrow[dr, shift left] \arrow[dl, shift left] & \\ \emptyset\ 0 \arrow[ur, shift left] \arrow[dr, shift left] & & \{x,y\}\ \mathbb{Z}/2\mathbb{Z} \arrow[ul, shift left] \arrow[dl, shift left] \\ & \{y\}\ \mathbb{Z}/2\mathbb{Z} \arrow[ur, shift left] \arrow[ul, shift left] & \end{tikzcd}$$

$X = \{x\} \cup \{y\}$

There are choices for s_x, s_y

If $s_x = 0$ $s_y = 1$, there is no $s \in \mathcal{F}(X)$ with $s|_{\{x\}} = s_x$, $s|_{\{y\}} = s_y$

<u>Defn</u>

If $\mathcal{F}, \mathcal{G}$ are (pre)sheaves on X, a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for every open set U s.t for every inclusion $V \subseteq U$

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r] \arrow[d, "\varphi(U)"'] & \mathcal{F}(V) \arrow[d, "\varphi(V)"] \\ \mathcal{G}(U) \arrow[r] & \mathcal{G}(V) \end{tikzcd}$$ commutes

An isomorphism is a morphism which has an inverse morphism

<u>Defn</u> Let $\mathcal{F}$ be a presheaf on a topological space X.

The stalk of $\mathcal{F}$ at $p \in X$ is

$$\mathcal{F}_p = \{(s, U) \mid s \in \mathcal{F}(U),\ U \text{ open containing } p\} / \sim$$

$(s, U) \sim (s', U')$ if $\exists$ open $V \subseteq U \cap U'$ with $s|_V = s'|_V$

ie $\mathcal{F}_p = \varinjlim_{U \ni p} \mathcal{F}(U)$

<u>Note:</u> $\mathcal{F}_p$ is a group.

$(s, U) + (s', U') = (s|_{U \cap U'} + s'|_{U \cap U'},\ U \cap U')$