

Introduction to schemes

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Recall: Affine varieties.

$\mathbb{A}^n = \mathbb{A}_k^n = K^n$ as a set, where k is a field.

$$\hat{A}_K = K^n$$

I ideal in $K[x_1, \dots, x_n]$,

$$V(I) = \{a \in \hat{A} \mid f(a) = 0 \forall f \in I\}$$

"the variety of I "

We give \hat{A} the Zariski topology
by making closed sets

$$\{V(I) : I \subseteq K[x_1, \dots, x_n]\}$$

The ideal of X is

$$I(X) = \left\{ f \in K[x_1, x_2] \mid f(a) = 0 \right. \\ \left. \forall a \in X \right\}$$

Hilbert's Nullstellensatz:

$$I(V(I)) = \sqrt{I} = \left\{ f \in K[x_1, x_2] : f^m \in I \right. \\ \left. \text{for some } m \geq 0 \right\}$$

Coordinate ring

$$K[X] = \frac{K[x_1, x_2]}{I(X)}$$

"polynomial functions
on X "

Affine varieties $/k \leftrightarrow$ f.g k -algebras

First description of this
module

with no
nilpotents

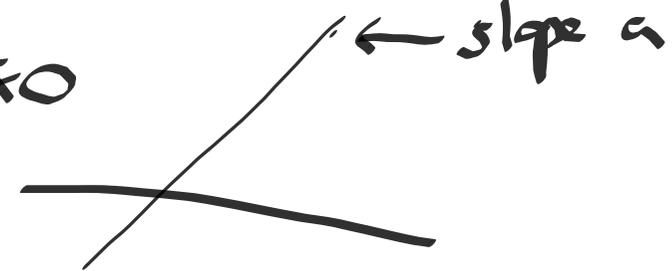
Answer the following questions:

1) What if we replace $I(X)$ by
an arbitrary ideal J ? (allow
nilpotents)

2) What if $K \neq \bar{K}$ or even K is not a field?

3) What about non-affine / non-projective varieties?

eg Consider $V(y(y-ax)) \subseteq \mathbb{A}^2$
for fixed $a \in K$. $a \neq 0$ ← slope a
 $a \rightarrow 0 \rightarrow V(y^2) = V(y)$ a double line



$$\text{eg } V(x^3 + y^3 - 1) \subseteq \mathbb{A}_{\mathbb{Q}}^2$$

$$(0, 1), (1, 0)$$

$$\text{eg } V(y(y - ax)) \subseteq \mathbb{A}_{k[a]}^2$$

.

Second description of module

A geometric space is a topological space with a notion of fns

eg $X = V(I)$, fns $k[X]$

eg X smooth manifold, fns analytic fns

eg X top manifold, fens cts fens

A scheme is a particular
choice of space and
of fens.

Refs: Hartshorne, Eisenbud-Harris,
Vakil's notes

Logistics:

If taking it for credit (or to get
on mailing list)
email me

For credit: Do an average of
3 HW Qs a week.

First topic: Sheaves

Let X be a topological space
I consider the set $\mathcal{O}(U)$
of cts fns $f: U \rightarrow \mathbb{R}$
for an open set U .

$$\mathcal{O}(U) = \{ \text{cts fns } X \rightarrow \mathbb{R} \}$$

Note: 1) For all $V \subseteq U$ open

we have the restriction map

$$\text{res}_{U,V}: \mathcal{O}(U) \rightarrow \mathcal{O}(V)$$

2) $W \subseteq V \subseteq U$ then $\mathcal{O}(U) \rightarrow \mathcal{O}(V)$

$$\text{res}_{U,W} = \text{res}_{V,W} \circ \text{res}_{U,V}$$

$$\begin{array}{ccc} \mathcal{O}(U) & \rightarrow & \mathcal{O}(V) \\ & \searrow & \downarrow \\ & & \mathcal{O}(W) \end{array}$$

3) If $U = \bigcup U_i$ is an open cover of U , and $f, g \in \mathcal{O}(U)$ with $\text{res}_{U, U_i}(f) = \text{res}_{U, U_i}(g) \quad \forall i$, then $f = g$.

~~4) If $U = \bigcup U_i$ is an open cover and $f_i \in \text{---}$.~~

4) If $U = \cup U_i$ is an open
 cover of U , and $f_i \in \mathcal{O}(U_i) \forall i$
 with $\text{res}_{U_i, U \cap U_j}(f_i) = \text{res}_{U_j, U \cap U_i}(f_j)$
 $\forall i, j$ then $\exists f \in \mathcal{O}(U)$ with
 $f_i = \text{res}_{U, U_i}(f)$.

\uparrow
 "agree on overlaps"

Defn Let X be a topological space

A presheaf \mathcal{F} of abelian groups on X consists of the data

a) For every open subset $U \subseteq X$ an abelian group $\mathcal{F}(U)$.

b) for every inclusion

$V \subseteq U$ of open subsets of X ,

a morphism of abelian groups

$$\rho_{uv}: \mathcal{F}(U) \rightarrow \mathcal{F}(V)$$

s.t. 0) $\mathcal{F}(\emptyset) = 0$

1) ρ_{uu} is the identity map
 $\mathcal{F}(U) \rightarrow \mathcal{F}(U)$.

2) If $W \subseteq V \subseteq U$, then

$$\rho_{uw} = \rho_{vw} \circ \rho_{uv}.$$

We call an element of $\mathcal{F}(U)$
a section of \mathcal{F} , and
 $\rho_{uv}(s)$ the restriction of s to v
— also write $s|_v$

Categorical Aside:

Let $\text{Top}(X)$ be the category whose objects are open sets and morphisms are inclusions
($\text{Hom}(U, V)$ is one element if $U \subseteq V$ and empty otherwise)

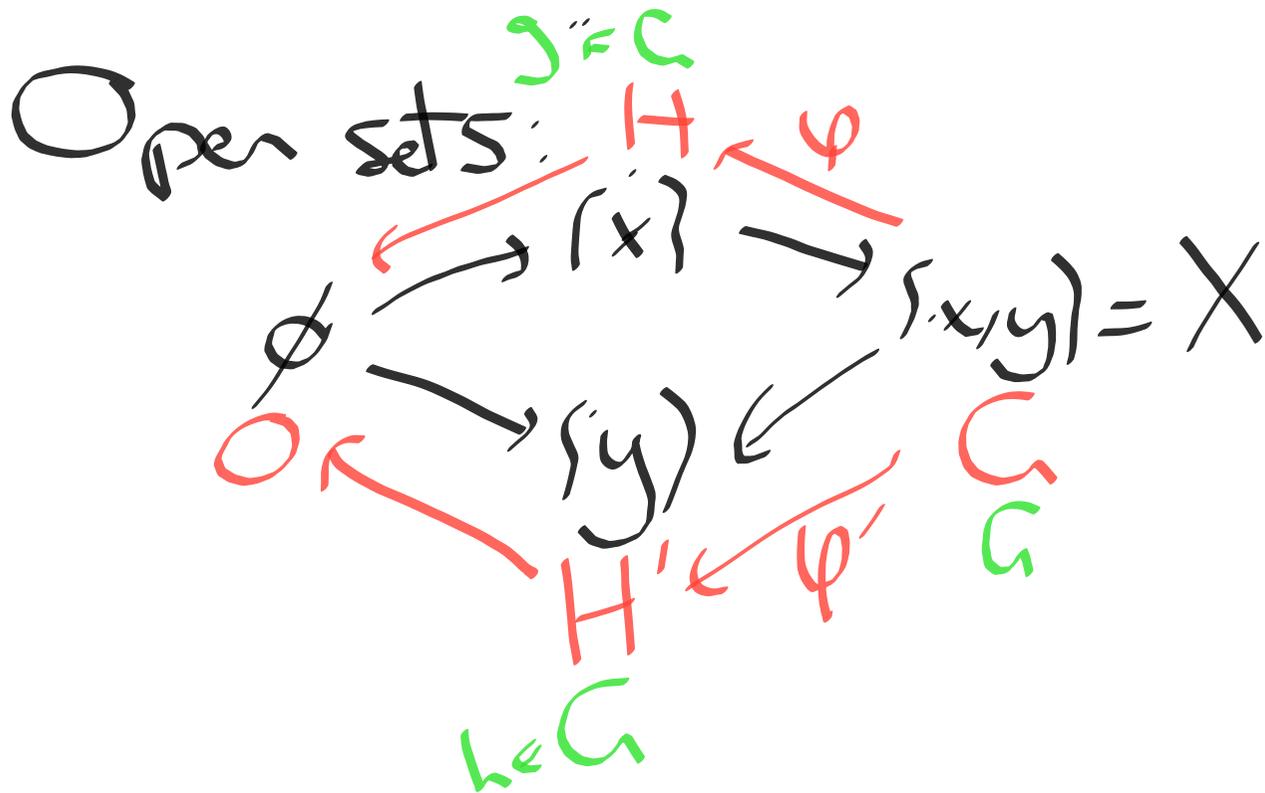
A presheaf is a functor
contravariant

$\text{Top}(X) \rightarrow \text{Abelian gps}$

(with $F(\emptyset) = \{0\}$.)

eg $X = \{x, y\}$ is the 2pt space

with the discrete topology



Defn A presheaf \mathcal{F} on a topological space X is a Sheaf if it satisfies the following two additional conditions:

3) IF U is an open set,
 $\{U_i\}$ is an open cover of U ,
and $s, s' \in \mathcal{F}(U)$ with $s|_{U_i} = s'|_{U_i}, \forall i$
Then $s = s'$.
(identity axiom)

4) If U is an open set and $\{U_i\}$ is an open cover of U , and $s_i \in \mathcal{F}(U_i)$ satisfy $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \quad \forall i, j$, then $\exists s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$. (gluing)

eg cts fns on X

There are presheaves that are not sheaves

eg fix an abelian group A

† set $\mathcal{F}(U) = A \quad \forall U \neq \emptyset$,

$\mathcal{F}(\emptyset) = \{0\}$ maps identity

ex: \mathcal{F} is a presheaf but not a sheaf. or \mathcal{O}

Defn If \mathcal{F}, \mathcal{G} , are

(pre)sheaves on X , a morphism

$\varphi: \mathcal{F} \rightarrow \mathcal{G}$ is a morphism

$\varphi(U): \mathcal{F}(U) \rightarrow \mathcal{G}(U) \quad \forall$ open set U

st whenever $V \subseteq U$

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow \rho_U & & \downarrow \rho_U \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

commutes

Defn: Let \mathcal{F} be a presheaf on a topological space X .

The stalk of \mathcal{F} at a pt $p \in X$ is $\{(s, U) \mid s \in \mathcal{F}(U), U \text{ open containing } p\}$
where $(s, U) \sim (s', U')$ if \exists open $V \subseteq U \cap U'$ with $s|_V = s'|_V$

This is the direct limit of
 the groups $\mathcal{F}(U)$ for U containing

\mathcal{P}

$$\text{ie } \varinjlim \mathcal{F}(U) = \bigcup_{U \ni \mathcal{P}} \mathcal{F}(U)$$

$$\begin{array}{l} s \sim s' \\ \text{if } \exists U \ni \mathcal{P} \\ \text{st } s|_U = s'|_U \end{array}$$

For \mathcal{F} the sheaf of cts fcs on X , the stalk of \mathcal{F} at p is the set of germs of fcs at p .

Note: The stalk of \mathcal{F} at p is a group.

$$(s, U) + (s', U') = (s|_{U \cap U'} + s'|_{U \cap U'}, U \cap U')$$

Check: This obeys the group axioms.

A morphism $\varphi: \mathcal{F} \rightarrow \mathcal{G}$

induces a homomorphism

$$\varphi_p: \mathcal{F}_p \rightarrow \mathcal{G}_p$$

$$\varphi_p \left(\underset{\mathcal{F}(U)}{\overset{\cap}{(s, U)}} \right) = \left(\underbrace{\varphi(U)(s)}_{\mathcal{G}(U)}, U \right)$$

Defn Let $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ be a
morphism of presheaves

The presheaf kernel of φ
is $U \mapsto \ker(\varphi(U))$

The presheaf cokernel is
 $U \mapsto \operatorname{coker}(\varphi) = \mathcal{G}(U) / \operatorname{im} \varphi$

presheaf image:

$$U \mapsto \text{im } \varphi(U)$$

If \mathcal{F}, \mathcal{G} are sheaves,

and $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ then the presheaf
kernel of φ is a sheaf. (check!)

Not true in general for image & cokernel.
(Soln: sheafify!)

A morphism $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ is an isomorphism if it has an inverse morphism.

see Hart. Prop 1.1 ch 2

~~Prop~~ Let $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ be a morphism of sheaves on X , then φ is an isomorphism if & only if the induced maps $\varphi_p: \mathcal{F}_p \rightarrow \mathcal{G}_p$ of stalks are isomorphisms $\forall p \in X$.

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