

Introduction to Schemes

Lecture 5

Last time: Properties of Schemes

- Connected
 - irreducible
 - reduced
 - integral
 - (locally) Noetherian
 - (locally) of finite type.
 - finite

- affine construction lemma
- open immersion
- closed embedding
- affine morphism

Today: More properties

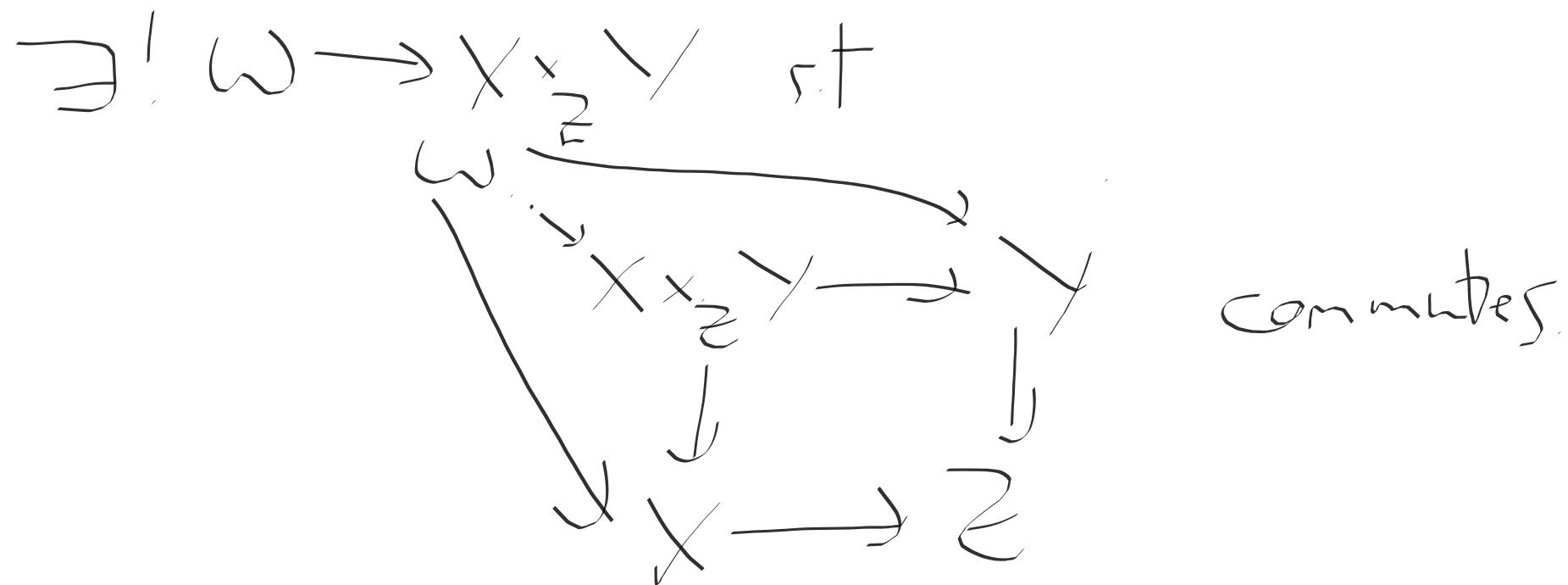
Fibre products, Separated, proper,
projective.

Recall: The product in a category is given by the following property: If

$\varphi: X \rightarrow Z$ then the (fibre) product

$X \times_Z Y$ is

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{\quad} & Y \\ \downarrow \varphi \quad \downarrow \psi & \curvearrowright & \downarrow \psi \\ X & \xrightarrow{\varphi} & Z \end{array}$$



eg When the category is Sets.

$$X \times_{\mathcal{Z}} Y = \{(a, b) \in X \times Y \mid \varphi(a) = \psi(b)\}$$

Products don't always exist in categories, but they do in the category of schemes.

"First case: everything is affine"

$$\begin{array}{ccc} \text{Spec}(A) & \xrightarrow{\psi} & Y = \text{Spec}(S) \\ \downarrow & \nearrow \psi^{-1} & \downarrow \\ \text{Spec}(R) & \xrightarrow{\quad} & Z = \text{Spec}(T) \end{array}$$

$$\begin{array}{ccccc} A & \xrightarrow{\psi} & S & & \\ \downarrow & & \downarrow & & \\ R \otimes T & \xrightarrow{\quad} & S & & \\ \downarrow & & \downarrow & & \\ R & \xleftarrow{\quad} & T & & \end{array}$$

$$X \times_Z Y = \text{Spec}(R \otimes_T S)$$

$$\begin{array}{c} R \otimes_T S \rightarrow A \\ r \otimes s \mapsto \psi(r)\psi(s) \end{array}$$

$$\begin{array}{c} R \rightarrow R \otimes_T S \\ r \mapsto r \otimes 1 \\ S \rightarrow R \otimes_T S \\ s \mapsto 1 \otimes s \end{array}$$

General case: This glues'
(see Hartshorne, Vakil, ...)

$$\text{eg } \mathbb{A}_K^n = \text{Spec}(K[x_1, x_n])$$

$$\mathbb{A}_K^n \times_K \mathbb{A}_K^m = \text{Spec}((K[x_1, x_n] \otimes_K K[y_1, y_m])^{\text{red}})$$

\parallel

$$\mathbb{A}^{n+m}$$
$$K[x_1, x_n, y_1, y_m]$$

Warning Topology on fibre
product is not the product

Topology

e.g. the topology on A^3
is not the product
from $A^1 \times A^1$.

Defn The residue field of
a pt on a scheme X

$$\text{is } k(P) = \mathcal{O}_P / \mathfrak{m}_P^T$$

Stalk
at P

maximal ideal
of local ring \mathcal{O}_P

If $X \rightarrow Y$ is a morphism of schemes, the fibre of $y \in Y$ is

$$X \times_Y \text{Spec}(k(y))$$

where $\text{Spec}(k(y)) \hookrightarrow Y$ comes from
 $\mathcal{O}(U) \rightarrow \mathcal{O}_y \rightarrow k(y)$
where U is an open containing y . (does not depend on choice)

$$\text{eg } \text{Spec}(\mathbb{C}[x,y]) \xrightarrow{\{(y^2 - x^3 + 2x)\}} \text{Spec}(\mathbb{C}[t])$$

$$\begin{matrix} 0 \\ \downarrow \\ t \end{matrix}$$

Fibre over $\langle x-1 \rangle$

$$\begin{aligned} K(\langle x-1 \rangle) &= \mathcal{O}_{\langle x-1 \rangle} / \langle x-1 \rangle \\ &= (\mathbb{C}[x])_{\langle x-1 \rangle} / \langle x-1 \rangle \cong \mathbb{C} \\ &\quad \begin{matrix} \nearrow & \searrow \\ x-1 & \end{matrix} \end{aligned}$$

So the fibre is

$$\text{Spec} \left(\frac{\mathbb{C}[x,y]}{(y^2 - x + 2x - 1)} \right) \cong \mathbb{C}[x]$$

$$\text{Spec} \left(\frac{\mathbb{C}[y]}{(y^2)} \right)$$

Fibre over $\langle 0 \rangle$

$$\begin{aligned} k(\langle 0 \rangle) &= \mathbb{C}(x) \\ &= \mathbb{C}(x)_{\langle 0 \rangle} / \langle 0 \rangle \end{aligned}$$

$$\begin{aligned} \mathbb{C}[x,y] &\quad \text{---} \\ \cancel{\langle y^2-x^3+2x-1 \rangle} &\quad \times \\ &\quad \mathbb{C}(x) \\ &\cong \mathbb{C}(x)[y] \cancel{\langle y^2-x^3+2x-1 \rangle} \end{aligned}$$

eg $X = \text{Spec}(\mathbb{Z}[x_1, x_n])$

$X \rightarrow Y = \text{Spec}(\mathbb{Z})$

(every scheme has a map to $\text{Spec}(\mathbb{Z})$)

The fibre over $\langle p \rangle$ is A^1_p

$\langle 0 \rangle$ is $A^1_{\langle 0 \rangle}$

Base change

If $\varphi: X \rightarrow Z$ is a morphism

$\psi: Y \rightarrow Z$ is another

then $X \times_Z Y \rightarrow Y$ is the
base change of φ along ψ

$$\begin{array}{ccc} & X \times_Z Y & \rightarrow X \\ \text{now} \quad \downarrow \theta & \downarrow & \downarrow \varphi \\ Y & \rightarrow Z & \end{array}$$

eg $Z = \text{Spec}(K)$
 $Y = \text{Spec}(K')$

$K' \not\subset K$
field extension

Many properties of morphisms
are "stable under base change"

meaning if φ had property P
then so does θ

e.g finite type $R \otimes_S T \xrightarrow{\quad} R$
Non-e.g integral $\text{Spec}(R[X]/(X^2+1)) \xrightarrow{\quad} \text{Spec}(R)$

This is integral, but the
base change over $\text{Spec}(\mathbb{C}) \rightarrow \text{Spec}(\mathbb{R})$

is not:

$$\text{Spec} \left(\frac{\mathbb{C}[x]}{(x^+)} \right) \rightarrow \text{Spec}(\mathbb{C})$$

Separated

Recall A topological space X is Hausdorff if $\forall u \neq v \in X$
there are disjoint open sets $U \ni u$,
 $V \ni v$, $U \cap V = \emptyset$.

\Rightarrow limits are unique.

Zariski topology
is almost never
Hausdorff!!

Ex A topological X is
Hausdorff iff the image of
 $\varphi: X \rightarrow X \times X^{\text{product topology}}$
 $x \mapsto (x, x)$ is closed.
in φ is called the diagonal.

Defn let $\varphi: X \rightarrow Y$ be a morphism of schemes.

The diagonal morphism

$$X \longrightarrow X \times X \quad \text{is}$$

the unique morphism coming from the identity morphisms $X \xrightarrow{\text{id}} X$



eg $X = \text{Spec}(R)$ $Y = \text{Spec}(S)$

$f: S \rightarrow R$

Δ $X \rightarrow X \times_Y X$

$R \leftarrow R \otimes_S R$

$ab \leftarrow a \otimes b$

Prop (Vakil 10.1.3)

The diagonal morphism

$X \rightarrow X \times_Y X$ is a locally closed embedding

$$\Delta = \tau \circ \rho \leftarrow$$

↑

open immersion

($\text{im } \Delta$ is
a closed subset
of an open set)
closed embedding

Defn A morphism $\varphi: X \rightarrow Y$

is separated if the diagonal

morphism is a closed embedding

(equiv (Hart Gr 4.2) if the image
of Δ is a closed subset of
 $X \times_Y X$)

Eg If $\varphi: X \rightarrow Y$ is a morphism
of affine schemes, then
 φ is separated.

$$\begin{array}{c} \triangle: \text{Spec}(R) \rightarrow \text{Spec}(R) \times_{\text{Spec}(S)} \text{Spec}(R) \\ R \otimes_S R \xrightarrow{\delta} R \\ a \otimes b \xrightarrow{\delta} ab \text{ is Surj } \text{Spec}(R \otimes_S R) \xrightarrow{\text{image is}} \text{Spec}(R) \end{array}$$

Eg let $X = "A"$ with the
origin doubled "

This has an open cover
 $X = U \cup V$ with $U, V \cong A'$,
glued along $A' \setminus \{0\}$.

~~Then X is not separated
over $\text{Spec}(k)$~~

$X \times_{\text{Spec}(k)} X$ is \mathbb{A}^2 with
doubled axes & 4 origins,
but the image of the diagonal only
includes 2 of the origins — all 4 are
in the closure

Properness

We've seen that affine schemes are (quasi)compact - but there feels to be something wrong in calling A^1 compact.

Recall: In topology a cts map
 $f: X \rightarrow Y$ is proper if the
preimage of a compact set is
compact (so if Y is a pt, if
 f is X is compact).

Defn A morphism is closed

if the image of any closed set
is closed

A morphism is universally closed
if $\begin{array}{ccc} Y' & \xrightarrow{\quad} & Y \\ X \times_{Y'} Y' & \xrightarrow{\quad} & Y' \end{array}$ the base change
 $X \times_Y Y' \xrightarrow{\quad} Y'$ is closed

eg $\varphi: A'_K \rightarrow \text{Spec}(k)$ is closed

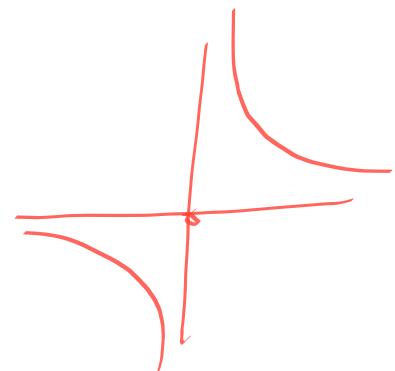
but the base change by

$A'_K \rightarrow \text{Spec}(k)$ is not

$\tilde{A} = A' \times_{\text{Spec}(k)} A' \xrightarrow{\text{Spec}(k(x))}$ is given by

$(x, y) \mapsto x$

$\text{Spec}(k(y)) / A' \rightarrow \text{Spec}(k)$ $V(xy - 1)$ has image $A'(0)$



Defn A morphism $\varphi: X \rightarrow Y$

is proper if it is separated,
of finite type, and universally
closed.

IF $Y = \text{Spec}(k)$ we also say
 X is complete