Intro to Schemes

Lecture 3

Last time: - Sheafification

- Image / cokernel of sheaves
- $\mathrm{Spec}(R)$ = set of prime ideals of R

Zariski topology: closed sets $V(I)$ $I \subseteq R$

Basis: $D(f) = \{ P : f \notin P \}$

Structure sheaf: $\mathcal{O}_{Spec(R)}$

$$\mathcal{O}_{Spec(R)}(D(f)) = R_f$$

$$= \left\{ \frac{a}{f^i} : a \in R, i \geq 0 \right\} / \sim$$

We had: If $D(f) \subseteq D(g)$ then $\exists\, m > 0$ s.t. $f^m \in \langle g \rangle$.

Thus if $D(f) \subseteq D(g)$, we can write $f^m = gr$ & get a homomorphism

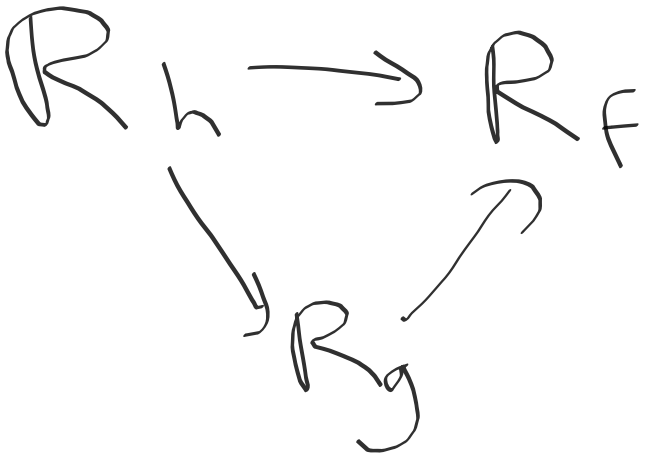
$$R_g \longrightarrow R_f$$

$$\frac{a}{g^n} \longrightarrow \frac{ar^n}{g^n r^n} = \frac{ar^n}{f^{mn}}$$

Well-defined: $\frac{a}{g^n} = \frac{a'}{g^{n'}}$, $f^{m'} = gr'$

Check: Restriction maps give a presheaf on the base:

$$D(f) \subseteq D(g) \subseteq D(h)$$



Now check identity + gluing:

Suppose $D(f) = \bigcup_{i \in J} D(f_i)$.

We claim $\exists m > 0$ s.t. $f^m \in \langle f_i : i \in J \rangle$

Otherwise by the lemma from last time:

says that an ideal max wrt containing $\langle f_i : i \in J \rangle$, and avoiding $U = \{1, f, f^2, f^3, \ldots\}$

would be prime, so $P \in D(f)$, but $P \notin D(f_i)$ for any i.

Since $D(f_i) \subseteq D(f)$ we can write $f_i^{m_i} = f r_i$ for some m_i, r_i $\forall i \in J$.

Identity: Suppose $\frac{a}{f^n} \in R_f$ satisfies $\frac{a r_i^n}{f_i^{mn}} = \frac{0}{1}$ in R_{f_i} $\forall i \in J$. Then $\exists N_i$ s.t. $f_i^{N_i} a r_i^n = 0$ in R. So $0 = f^n f_i^{N_i} a r_i^n = f_i^{N_i + mn} a$

$f_i^{N_i+m_i} a = 0$.

Since $f^m \in \langle f_i : i \in J \rangle$ we can find

$N \gg 0$ s.t $f^N \in \langle f_i^{N_i+m_i} : i \in J \rangle$

$$f^N = \sum_{i \in J} b_i f_i^{N_i+m_i} \qquad |J| < \infty$$

Thus $f^N a = \sum b_i f_i^{N_i+m_i} a = 0$,

and so $\frac{a}{f^n} = 0$ in R_f.

Claim Write $f^m = \sum_{i \in J} b_i f_i$ $|J| < \infty$

Then $D(f) = \bigcup_{j \in J} D(f_j)$ as if

$f_i \in P$ $\forall i \in J$ then $f \in P$

Suppose we are given $s_i = \frac{a_i}{f_i^{n_i}} \in R_{f_i}$

s.t. ☆ $\frac{a_i f_j^{n_i}}{(f_i f_j)^{n_i}} = \frac{a_j f_i^{n_j}}{(f_i f_j)^{n_j}} \in R_{f_i f_j}$ $\forall i,j \in J$.

It suffices to show there is

$s \in R_f$ with $s|_{D(f_i)} = s_i$ for $i \in \beta$.

Identity will then imply $s|_{D(f_i)} = s_i$ $\forall i \in \beta$

Since β is finite we may assume that $n_i = N$ for all $i \in \beta$.

Then $\otimes$ says $(f_i f_j)^N (a_i f_j^N - a_j f_i^N) = 0$ $\forall i,j \in \beta$

$$a_i f_i^{\tilde{N}} f_j^{\tilde{N}+N} - a_j f_j^{\tilde{N}} f_i^{\tilde{N}+N} = 0$$

So after renaming $a_i = a_i f_i^{\tilde{N}}$ $\forall i \in \mathcal{I}$,

$$a_i f_j^{N'} = a_j f_i^{N'}$$

Write $f^M = \sum_{i \in \mathcal{I}} c_i f_i^{N'}$ for some $M > 0$.

Set $g = \sum_{i \in \mathcal{I}} c_i a_i$. Claim: $\left.\frac{g}{f^m}\right|_{D(f_i)} = s_i$

Indeed, $\left.\frac{g}{f^m}\right|_{D(f_i)} = \frac{g r_i^m}{f_i^{m \cdot m}}$ $\left(f^m = g r_i\right)$

$$\Rightarrow f_i^{N'} g = \sum_{j \in \mathbb{N}} f_i^{N'} c_j a_j = \sum_{j \in \mathbb{N}} c_j a_i f_j^{N'} = a_i f^M$$

$$\Rightarrow f_i^{N' M} r_i^m g = a_i r_i^m f^M = a_i f_i^{m \cdot M}$$

$$\text{so } \frac{g r_i^M}{f_i^{m \cdot M}} = \frac{a_i}{f_i^N} = s_i \text{ in } R_{f_i} \text{ as required}$$

We thus have a sheaf on the base $\{D(f)\}$, so get a sheaf $\mathcal{O}_{\mathrm{Spec}(R)}$ on $\mathrm{Spec}(R)$ from this.

Prop a) The stalk $(\mathcal{O}_{Spec(R)})_p$ of $\mathcal{O}_{Spec(R)}$ at a prime p is the localization $R_p = \left\{ \frac{r}{f} : f \notin p \right\}$

b) $\mathcal{O}_{Spec(R)}(U) = \{ \text{fcns } s : U \to \coprod_{p \in U} R_p : \forall p \in U, s(p) \in R_p \text{ and } \forall p \in U, \exists V \subseteq U \text{ an open, } f \in R \text{ s.t. } \forall Q \in V, f \notin Q, \text{ and } s(Q) = \frac{g}{f} \text{ in } R_Q \}$

Pf of a)

The stalk $\mathcal{F}_p$ of a sheaf $\mathcal{F}$ coming from a sheaf on the base is $\{(s, B) \mid s \in \mathcal{F}(B), p \in B\} / \sim$

For $P \in \mathrm{Spec}(R)$ we have $P \in D(f)$ if & only if $f \notin P$. Construct

$\varphi : R_p \longrightarrow (\mathcal{O}_{\mathrm{Spec}(R)})_P$ by

$\varphi : R_p \longrightarrow (\mathcal{O}_{Spec(R)})_p$ by

$$\varphi\left(\frac{r}{f}\right) = \left(\frac{r}{f}, D(f)\right)$$

Check: 1) well-defined

2) φ is a ring homomorphism

3) φ is surj. (if $(s, D(f)) \in (\mathcal{O}_{Spec(R)})_p$ then $s = \frac{r}{f^m}$ for some $r \in R$, $m \geq 0$, $f \notin p$,

so $\frac{r}{f^m} \in R_p$ & $\varphi\left(\frac{r}{f^m}\right) = (s, D(f^m)) = (s, D(f))$

4) φ is inj.

b) is the statement of how to recover a sheaf from a sheaf on the bas.

Note: This means that the stalk of $\mathcal{O}_{Spec(R)}$ at a pt P is a local ring (unique max ideal)

Morphisms

Defn Let $f: X \to Y$ be a cts map of topological spaces & let $\mathcal{F}$ be a sheaf on X. The direct image sheaf $f_* \mathcal{F}$ on Y is given by $f_* \mathcal{F}(U) = \mathcal{F}(f^{-1}(U))$

Defn A ringed space is a pair $(X, \mathcal{O}_X)$ of a topological space X & a sheaf of rings $\mathcal{O}_X$ on X.

A morphism of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ of a cts map $f: X \to Y$

and a map $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$

of sheaves of rings on Y.

eg A ring homomorphism $\varphi : R \to S$

induces

$$\varphi : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$$

$$P \longmapsto \varphi^{-1}(P) \leftarrow \text{check prime.}$$

$gh \in \varphi^{-1}(P)$, $\varphi(gh) \in P$, so $\varphi(g)\varphi(h) \in P$

so $\varphi(g) \in P$ or $\varphi(h) \in P$, so $g \in \varphi^{-1}(P)$ or $h \in \varphi^{-1}(P)$.

φ is cts:

If $V(I)$ is a closed set in $\mathrm{Spec}(R)$,

then $\varphi^{-1}(V(I)) = \{ P \in \mathrm{Spec}(S) \mid \varphi^{-1}(P) \in V(I) \}$

$= \{ P \in \mathrm{Spec}(S) \mid I \subseteq \varphi^{-1}(P) \}$

$= \{ P \in \mathrm{Spec}(S) \mid \varphi(I) \subseteq P \}$

$= V(\varphi(I))$.

So inverse images of closed sets are closed.

Ex: $\varphi^{-1}(D(f)) = D(\varphi(f))$

Define $\varphi^{\#} : \mathcal{O}_{Spec(R)} \to \varphi_* \mathcal{O}_{Spec(S)}$

on $D(f)$ by $\varphi^{\#}(D(f))\left(\frac{a}{f}\right) = \frac{\varphi(a)}{\varphi(f)} \in S_{\varphi(f)}$

This suffices to define $\varphi^{\#}$

(eg ex 2.7.C Vakil, Prop I.12 Eisenbud-Harris)

Defn A ringed space $(X, \mathcal{O}_X)$ is a locally ringed space if $\forall P \in X$ the stalk $\mathcal{O}_{X,P}$ is a local ring

eg $(Spec(R), \mathcal{O}_{Spec(R)})$

A morphism $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ induces a ring homomorphism

$$f_p^{\#} : \mathcal{O}_{Y, f(p)} \longrightarrow \mathcal{O}_{X,p}$$

by $(s, U) \longmapsto \left(f^{\#}(U)(s), f^{-1}(U)\right)$

$$f^{\#}(U) : \mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$$

A morphism of locally ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism $(f, f^\#)$ of ringed spaces s.t $\forall p$, $f_p^\#$ is a local homomorphism ie $f_p^{\#-1}(\text{max ideal}) = \text{max ideal}$

$m_{Y,f(p)}$ $m_{X,p}$

(always have $\subseteq$)

eg for $\varphi : R \to S$

$(\varphi, \varphi^{\#})$ is a morphism of locally ringed spaces, &

$\varphi_p^{\#} : R_{\varphi^{-1}(p)} \longrightarrow S_p$ is given by

$$\frac{a}{f} \longmapsto \frac{\varphi(a)}{\varphi(f)}.$$

In fact (Hartshorne Prop 2.3(c))

Morphisms of locally ringed spaces $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$

$\Updownarrow$

ring homomorphisms $R \to S$.

Defn An affine scheme is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic (as a locally ringed space) to $\mathrm{Spec}(R)$ for some R.

(isomorphism = $\exists$ inverse)

A scheme is a locally ringed space $(X, \mathcal{O}_X)$ for which every pt has an open nbhd U s.t $(U, \mathcal{O}_X|_U)$ is an affine schem

↑ restriction to U.

We call X the underlying space, $\mathcal{O}_X$ the structure sheaf.

A morphism of schemes is a morphism of locally ringed spaces. An isomorphism is a morphism with a two-sided inverse.

eg affine varieties "are"

schemes $\mathrm{Spec}(R)$ where

R is a finitely gen alg over a field with no nilpotents

coordinate ring of X.

eg not all schemes are affe.

$A^2 \setminus \{0\}$. Fix a field

k, & let $R = k[x, y]$.

Set $X = \mathrm{Spec}(R) = \mathbb{A}^2$

Set $U = D(x) \cup D(y) = \mathrm{Spec}(R) \setminus \{(x, y)\}$

Consider $Y = (U, \mathcal{O}_{X|U})$

This is a scheme, but is not affine.

Claim: $\mathcal{O}_Y(Y) = K[x,y]$

If Y were affine, it would be isomorphic to $\operatorname{Spec}(K[x,y])$.