

Introduction to Schemes

Lecture 2

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Last time:

- Introduction to module.

- Sheaves:

Presheaf \mathcal{F} on a topological space X is a group $\mathcal{F}(U)$ for every open set U on X & $\forall V \subseteq U$ $\rho_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$

s.t. $w \in V \subseteq U$ & $\rho_{WW} = \text{identity}$
 $\mathcal{F}(U) \xrightarrow{\rho_{UV}} \mathcal{F}(W) \xrightarrow{\rho_{WW}} \mathcal{F}(U)$
 $\rho_{UU} = \text{identity}$
 $\mathcal{F}(\emptyset) = \{0\}$

Sheaf: add

identity: $U = \bigcup U_i$, if $\rho_{U_i}(s) = \rho_{U_i}(s')$ then $s = s'$

Gluing: Given $s_i \in \mathcal{F}(U_i)$ with

$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ then $\exists s \in \mathcal{F}(U)$
with $s|_{U_i} = s_i$

Morphisms of (pre)sheaves:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \text{Res}_U \downarrow & \curvearrowleft & \downarrow \text{Res}_U \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

Stalk: $\mathcal{F}_p = \lim_{U \ni p} \mathcal{F}(U) = \coprod_{U \ni p} \mathcal{F}(U)$ / ~~$s \sim s'$
if $\exists V \ni p$
 $s \upharpoonright V = s' \upharpoonright V$~~

$\psi: \mathcal{F} \rightarrow \mathcal{G}$ induces $\psi_p: \mathcal{F}_p \rightarrow \mathcal{G}_p$

Announcement:

- Email D.MacLagan@warwick.ac.uk to get on class list.
- When emailing HW
make filename
TCCSthesesHW/Lastname.pdf

Sheafification

Every presheaf has an associated sheaf.

Defn Let \mathcal{F} be a presheaf on a topological space X . Define a presheaf \mathcal{F}^+ on X by:

$\mathcal{F}^+(U) = \{ \text{Funcs } s: U \rightarrow \bigsqcup_{P \in U} \mathcal{F}_P \text{ s.t.}$

$s(p) \in \mathcal{F}_p \quad \forall p \in U \text{ and}$

$\forall p \in U \exists \text{ open } V \subseteq U$

$\exists t \in \mathcal{F}(V) \text{ s.t. } \forall q \in V$

$s(q) = t_q \leftarrow \text{gen of } t \text{ at } Q \}$

$(t, V) \in \mathcal{F}_q.$

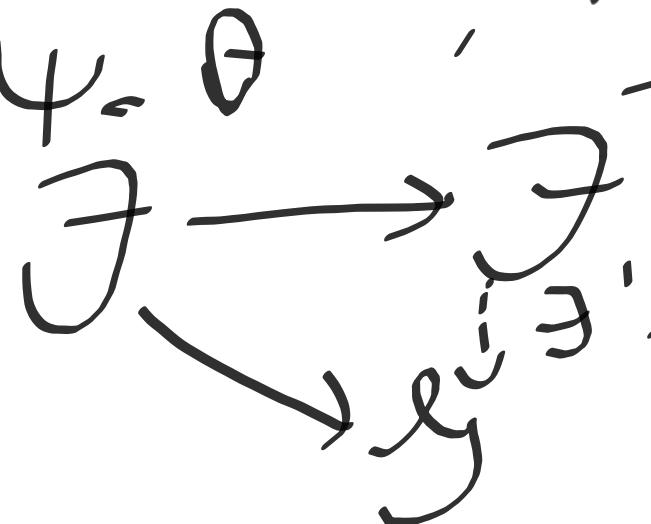
Exercise 1). \mathcal{F}^+ is a sheaf

2) There is a natural morphism
of presheaves $\theta: \mathcal{F} \rightarrow \mathcal{F}^+$

$$s \in \mathcal{F}(U) \mapsto (s^+: U \rightarrow \bigcup \mathcal{F}_p)$$
$$s^+(\rho) = s_\rho$$

3) For any sheaf \mathcal{Y} and
morphism of presheaves

$\varphi: \mathcal{F} \rightarrow \mathcal{Y}$ there is a
unique morphism $\psi: \mathcal{F}^+ \rightarrow \mathcal{Y}$
s.t $\varphi = \psi \circ \theta$



Ex: We saw last time that
the constant presheaf on the
two pt space X is not a sheaf

$$\begin{array}{ccc} & \overset{\text{A}}{\underset{(x)}{\leftarrow}} & \\ \overset{\text{B}}{\leftarrow} & & \overset{(x,y)}{\leftarrow} \\ \text{O} & \overset{\text{Y}}{\leftarrow} & \text{A} \\ \text{D} & \overset{\text{A}}{\leftarrow} & \text{A} \end{array} \quad \begin{array}{l} \mathcal{F}_x = A \\ \mathcal{F}_y = A \end{array}$$

Note: 3) applied to

$\text{id}: \mathcal{F} \rightarrow \mathcal{F}$ shows that
if \mathcal{F} is a sheaf then
 $\mathcal{F} \approx \mathcal{F}^+$.

Defn let $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ be a
morphism of sheaves.

The sheaf image of φ is
the sheafification of the presheaf
image of φ .

The sheaf cokernel is the sheafification
of the presheaf cokernel of φ .

$\varphi: \mathcal{D} \rightarrow \mathcal{E}$ is injective

if $\ker \varphi = \emptyset$

& say if $\text{im } \varphi \simeq \mathcal{E}$

Affine Schemes

Let R be a commutative ring with identity.

We now define the affine scheme $\text{Spec}(R)$.

Defn ($\text{Spec}(R)$ as a topological space)

The set $\text{Spec}(R)$ is the set of all prime ideals in R .
 $(f, g \in R \Rightarrow f, g \in P \Rightarrow fg \in P)$

We place the Zariski topology on $\text{Spec}(R)$

Closed sets: $V(I) = \{P \subseteq R \text{ s.t. } I \subseteq P\}$
for all ideals $I \subseteq R$.

Ex: Check that this is a topology.

$$V(I \cap J) = V(I) \cup V(J)$$
$$V(I + J) = V(I) \cap V(J)$$

eg $R = \mathbb{Z}$ $\text{Spec}(\mathbb{Z}) = \{0\} \cup \{p\mathbb{Z}\}$
 p prime

$p\mathbb{Z}$ are closed points.

$$\overline{\{0\}} = \text{Spec}(\mathbb{Z})$$



eg $R = k[x]$. $\text{Spec}(R) = \{\text{prime ideals in } R\}$

$$K = \overline{k} \quad \{0\}/(kx-a) : a \in K\}$$

$$K = \mathbb{R} \setminus \{0\} \cup \{x - a : a \in \mathbb{R}\}$$

$$\cup \left\{ x^2 + ax + b : a^2 - b < 0 \right\}$$

$$K = \mathbb{Z}/2\mathbb{Z} \setminus \{0\}, \{x\}, \{x+1\}, \dots$$

(check there are infinitely many irreducible polys.)

e.g $R = k[x_1, x_n]$, $k \in \mathbb{F}$

closed pts \hookrightarrow maximal ideals

Nullstellensatz in R

$$\hookrightarrow \langle x_i - a_i : 1 \leq i \leq n \rangle$$

\hookrightarrow pts in $\mathbb{A}^n_{\text{naive}}$

Every irreducible subvariety of \mathbb{A}^n gives
another pt of $\text{Spec}(R)$

For every nonzero $f \in R$ we have

the basic open set $D(f)$

$$= \text{Spec}(R) \setminus V(f)$$

$$= \{P \in \text{Spec}(R) \mid f \notin P\}$$

The set of all $D(f)$ form a basis for the Zariski topology on $\text{Spec}(R)$.

Structure Sheaf

This will be a sheaf of rings.

HW1 Q4 Looks at "sheaves
on a base"

If \mathcal{B} is a basis for the topology
on X & $\{\mathcal{I}(B) : B \in \mathcal{B}\}$ is a

collection of groups (rings)
satisfying the sheaf axioms
with respect to \mathcal{B} (i.e,

open covers are $\mathcal{B} = \cup \mathcal{B}_i$, $\mathcal{B}, \mathcal{B}_i \in \mathcal{B}$,
then \exists^{sheaf} \mathcal{F} on X with given values
for $\mathcal{F}(\mathcal{B})$.

We'll define the structure sheaf

$\mathcal{O}_{\text{Spec}(R)}$ on $\text{Spec}(R)$

by setting

$$\mathcal{O}_{\text{Spec}(R)}(D(f)) = R_f$$

$\frac{\{f_i : f_i \in R\}}{\{f_i : f_i \neq 0\}}$ = R localized at the multi closed set $\{1, \epsilon, f, f^2, \dots\}$

We now check that $\mathcal{O}_{\text{Spec}(R)}$
is a sheaf on the base.

Lemma If I is an ideal of R disjoint from a multiplicatively closed set U & P is an ideal of R maximal wrt containing I & disjoint from U , then P is prime.

Pf Suppose $f, g \notin P$. Then $P + \langle f \rangle$,
 $P + \langle g \rangle$ are strictly large ideals containing
 I . Thus $\exists u_1, u_2 \in U$ ($r_1, r_2 \in R$),
 $P_1, P_2 \subset P$ with $u_1 = P_1 + r_1 f$, $u_2 = P_2 + r_2 g$
 Since $u_1, u_2 \in U$, $\underbrace{P_1 P_2 + R(r_2 g + P_2 r_1 f + r_1 r_2 f g)}_{\in P}$
 so $r_1 r_2 f g \in P$, thus $f g \in P$ $\notin P$

COR If $D(f) \subseteq D(g)$ then

$\exists n > 0$ s.t. $f^n \in \langle g \rangle$

Pf If not, $U = \{1, f, f^2, \dots\}$ is disjoint from $I = \langle g \rangle$, so by the lemma, there is a prime ideal P with $g \in P$, $P \cap U = \emptyset$, so $D \in D(f) \setminus D(g)$.

Thus if $D(f) \subseteq D(g)$, we have

$f^m = g^r$ for some $m > 0, r \in \mathbb{R}$.

Define

$$\rho_{D(g), D(f)} : R_g \rightarrow R_f$$

$$\begin{array}{ccc} \frac{a}{g^r} & \mapsto & \frac{a^m}{g^{mr}} = \frac{a^r}{f^{mn}} \\ R_h \rightarrow R_f & \nearrow & \downarrow R_g \end{array}$$

This satisfies
 $D(f) \subseteq D(g) \subseteq D(h)$

(well-defined 1) $R_g \rightarrow R_f$

$$f^m = g^r$$

$$g^{\frac{a}{r}} \downarrow \frac{ar^m}{f^m}$$

Suppose $\frac{a'}{g'} = \frac{a}{g}$. Then \exists also s.t

$$g'(ag - ag') = 0$$

$$(f^m a^r - f^m a^r') = 0$$

2) What if $f^m = g^r$ $f^{m'} = g^{r'}$
 $f^{m-m} = g^{r'-r}$