INTRODUCTION TO SCHEMES - HOMEWORK 1

DIANE MACLAGAN

This is too many exercises for almost everyone taking this module (with the exception of 1st year algebraic geometry students whose supervisors have told them to focus on learning schemes this term). Focus on the questions roughly in order. It is worth reading all the questions from Hartshorne before attempting any; a lot of the content in Hartshorne is in the exercises. The reference to Vakil's notes is to the June 4, 2017 version.

Not all of these exercises produce something suitable for handing in; if you are taking this module for credit, these do not count towards your list of homework assignments.

If you think you have discovered a mistake in one of these exercises, please email me as soon as possible.

(1) (Review) If you have not had a first course in algebraic geometry (eg equivalent to Warwick's MA4A5) get a textbook for such a course, and read some of it. Accessible choices include (but are very much not limited to) Reid's "Undergraduate Algebraic Geometry", and Hassett's "Introduction to Algebraic Geometry". Ask your supervisor for their suggestions (you may want to phrase this as asking for their favourite book to read before Hartshorne).

(2) (Review) If you have not had a course in commutative algebra (and recommended even if you haven't) get a copy of a commutative algebra book that can you can refer to during the term.

(3) Vakil 2.2.G. Let $\phi\colon X \to Y$ be a continuous map of topological spaces. Show that sections of ϕ form a sheaf. More precisely, for each open set $U \subset Y$, set $\mathcal{F}(U)$ equal to the set of continuous maps $s : U \to X$ such that $\phi \circ s = \mathrm{id}|_U$. Show that $\mathcal{F}$ is a sheaf on Y.

(4) Recall that a base for a topological space X is a subset of open sets $\mathcal{B}$ such that every open set $U \subset X$ is a union of sets in $\mathcal{B}$. Show that if $\mathcal{F}$ is a sheaf, then $\mathcal{F}$ is determined by the subsets $\mathcal{F}(B)$ for $B \in \mathcal{B}$, and morphisms $\mathcal{F}(B) \to \mathcal{F}(B')$ for $B' \subset B$, $B, B' \in \mathcal{B}$. Read and understand Theorem 2.7.1 of Vakil's notes. (See also Section I.1.3 of Eisenbud-Harris).

(5) Fix an algebraically closed field K. Let X be $\mathbb{A}^n = \mathbb{A}^n_K$ with the Zariski topology. As a set, $\mathbb{A}^n$ is K^n. Closed sets in $\mathbb{A}^n$ are subvarieties $V(I) = \{x \in \mathbb{A}^n : f(x) = 0 \text{ for all } f \in I\}$ for an ideal $I \subset K[x_1, \ldots, x_n]$.
   (a) Show that the sets $D(f) = \mathbb{A}^n \setminus V(f)$ for $f \in K[x_1, \ldots, x_n]$ form a basis for the Zariski topology.
   (b) Define a sheaf $\mathcal{O}$ on $\mathbb{A}^n$ as follows. For $D(f)$ set $\mathcal{O}(D(f))$ to be the localization $K[x_1, \ldots, x_n]_f$. Verify that this is enough information to define a sheaf by checking that it satisfies the conditions for Theorem 2.7.1 of Vakil's notes (you will also have define the restriction homomorphisms).

(6) Hartshorne Q1.1. (sheafification of the constant presheaf)

(7) Hartshorne Q1.17. (Skyscraper sheaves)

(8) an example of an exact sequence

(9) Hartshorne Q1.2. (exactness and stalks)

(10) Hartshorne Q1.3. (surjectivity)

(11) Hartshorne Q1.5 (characterization of isomorphism)

(12) Hartshorne Q1.6 (exact sequences behave the way you think they do)

(13) Hartshorne Q1.14. (Support of a sheaf)

(14) Hartshorne Q1.22. (Gluing sheaves)

(15) Do the rest of the exercises in Hartshorne Chapter 2, section 1!