TCC Hilbert schemes and Moduli Spaces

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No lecture next week (1 November)
Possibly also not 8 November

Classification For a moduli functor
$F : \underline{\text{Schemes}} \to \underline{\text{Sets}}$ with $F(B) = \{\text{families over } B \text{ st } \cdots\}/\sim$
For $\varphi : B \to B'$ $F(\varphi) : F(B') \to F(B)$
is (usually) pullback

$\varphi : B \to B'$ $\quad F(\varphi)\begin{pmatrix} Y \\ \downarrow \\ B' \end{pmatrix} = \begin{pmatrix} \varphi^* Y \\ \downarrow \\ B \end{pmatrix}$

$$\begin{tikzcd} \varphi^* Y \arrow[r] \arrow[d] & Y \arrow[d] \\ B \arrow[r] & B' \end{tikzcd}$$

Today Construction of the Hilbert scheme

Recall $\mathrm{Hilb}_p(\mathbb{P}^n)(B) = \{$ flat families over B of subschemes of $\mathbb{P}^n_B$ with Hilbert poly of fibres equal to $P\}$

We want to show this is representable.

Key idea: Find a degree D, depending only on P with

1) Every saturated ideal I with Hilbert polynomial P is generated in degree at most D, so $I_{\geq D} = \langle I_D \rangle$

2) If I is an ideal generated in degree D and has $h_I(d) = P(d)$ for $d = D, D+1$, then I has Hilb. poly P.

Consider $I_D \in Gr\left(\binom{n+D}{n} - P(D), S_d\right)$ $k[x_0, \ldots x_n]$

We then get $\mathrm{Hilb}_P(\mathbb{P}^n)$ as a subscheme of $\mathrm{Gr}\left(\binom{n+D}{n} - P(D), \binom{n+D}{n}\right)$

with equations coming from checking that $\langle I_D \rangle$ has the correct Hilbert fcn in degree $D+1$.

Castelnuovo-Mumford regularity

This is an invariant of $Z \subseteq \mathbb{P}^n$ introduced by Mumford

Defn Let Z be a subscheme of $\mathbb{P}^n$. The regularity of Z is (Exists by Serre vanishing)

$$\operatorname{reg}(Z) = \min\{ j : H^i(\mathbb{P}^n, \mathcal{O}_Z(l-i)) = 0 \text{ for all } l \geq j,\ i > 0 \}$$

Geometric defn

sheaf cohomology

Algebraic defn

If $I \subseteq K[x_0, \dots, x_n]$ is the homogeneous saturated ideal corresponding to Z,

$$\operatorname{reg}(Z) = \operatorname{reg}(S/I) = \min\left\{ j : H^i_m(S/I)_\ell = 0 \text{ for all } \ell > j-i,\ i \geq 0 \right\}$$

local cohomology

Eisenbud-Goto

$$= \max_{i,j} (\beta_{ij} - i)$$

where $0 \leftarrow S/I \leftarrow F_0 \leftarrow \dots \leftarrow F_s \leftarrow 0$ is the minimal free resolution of S/I where $F_i = \bigoplus_j S(-\beta_{ij})$

shift grading

$S(-a)_b = S_{b-a}$

Key facts 1) The β_{ij} are the degrees of generators of I, so since $\operatorname{reg}(S/I) = \max(\beta_{ij} - i) \geq \beta_{1j} - 1$

$\operatorname{reg}(S/I) + 1$ is an upper bound for the degrees of generators of I.

2) (If I is saturated), then $h_I(d) = P_I(d)$ for $d \geq \operatorname{reg}(S/I) + 1$.

We will give a uniform bound on the regularity of all saturated ideals with Hilbert polynomial P

Sample question Is there a subscheme $Z \subseteq \mathbb{P}^n$ with Hilbert polynomial t^2 for some $\mathbb{P}^n$?

Theorem (Macaulay)

The Hilbert polynomial of a homogeneous ideal $I \subseteq k[x_0, \ldots, x_n]$ can be written as

$$p(t) = \sum_{i=1}^{D} \binom{t+a_i-i+1}{a_i} = \binom{t+a_1}{a_1} + \binom{t+a_2-1}{a_2} + \cdots$$

where $a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_D \geq 0$

$$\binom{t+2}{2} = \frac{1}{2}(t+2)(t+1)$$

$$p(t) = \sum_{i=1}^{D} \binom{t + a_i - i + 1}{a_i}$$

So no for t^2! $\binom{t+a_i-i+1}{a_i}$ has degree a_i.

So must have $a_1 = 2$.

$t^2 - \binom{t+2}{2}$ has degree 1, so $a_2 = 2$,

but $\binom{t+2}{2} + \binom{t+1}{2} = (t+1)^2 > t^2$

So can't keep going

$\mathrm{Hilb}_{t^2}(\mathbb{P}^n) = \emptyset$ for all n.

Theorem cont. If J is a homogeneous ideal with $h_J(d) = p(d)$, then

$$h_J(d+1) \leq p(d+1)$$

"Hilbert fcn can't grow faster than Hilb poly"

The expression for the Hilbert polynomial comes from considering the lexicographic ideal

Defn $x^{\underline{u}} < x^{\underline{v}}$ if the first nonzero entry of $\underline{v} - \underline{u}$ is positive.

$x_0^{u_0} x_1^{u_1} \cdots x_n^{u_n}$

$x_0 > x_1 > x_2 > \cdots$, $\quad x_1^2 > x_2^3$

The lexicographic ideal with Hilbert function h is the monomial ideal with

I_{lex} $\qquad h(d) = \dim (S/I)_d$

$(I_{lex})_d$ spanned by the first (largest) $\qquad \dim S_d = \binom{n+d}{d}$

$\dim I_d \rightarrow \binom{n+d}{d} - h(d)$ monomials wrt $<$

Macaulay proves this always exists $\quad S_1 (I_{lex})_d \subseteq (I_{lex})_{d+1}$

Theorem (Gotzmann)

Let $I \subseteq K[x_0, \dots, x_n]$ be a homogeneous ideal, saturated with respect to $\langle x_0, \dots, x_n \rangle$.

Write $p_I(t) = \sum_{i=1}^{D} \binom{t+a_i-i+1}{a_i}$ with $a_1 \geq a_2 \geq \dots \geq a_D \geq 0$.

Then $\operatorname{reg}(S/I) \leq D-1$.

In particular, I is generated in deg $\leq D$.

Computable uniform bound on regularity for all saturated ideals with Hilbert poly P.

Consequence All saturated ideals with Hilbert polynomial P are generated in degree $\leq D$ (so $I_{\geq D} = \langle I_D \rangle$) and $h_I(d) = p_I(d)$ for $d \geq D$

$\rightsquigarrow$ $I_D \in Gr\left(\binom{n+D}{n} - p(D), S_D\right)$

$\uparrow$ $\dim \binom{n+D}{n}$

The number D is called the Gotzmann number of P.

Theorem (Gotzmann) Persistence thm

Let P be a Hilbert polynomial.
If I is a homogeneous ideal with
$h_I(d) = p(d)$ and $h_I(d+1) = p(d+1)$
then $h_I(m) = p(m)$ for $m \geq d$, so
I has Hilbert polynomial p.
We will apply this for $d = D$.

$\Rightarrow (I : \langle x_0, \dots, x_n \rangle^\infty)$ has Hilbert polynomial p,
so is generated in degrees $\leq D$, so $I_{\geq D} = (I : \langle x_0, \dots, x_n \rangle^\infty)_{\geq D}$

So we have a bijection between

$\{$ homogeneous saturated ideals in $K[x_0, \ldots, x_n]$ with Hilbert poly $P \}$

and

$\{$ pts $p \in Gr\left(\binom{n+D}{n} - p(D), S_D\right)$ for which ideal gen by $\langle p \rangle$ has Hilbert fun

$h_{\langle p \rangle}(D) = p(D), \; h_{\langle p \rangle}(D+1) = p(D+1)$.

Extends in a natural way to R replacing K {pts $p \in Gr\left(\binom{n+D}{n} - p(D), \binom{n+D}{n}\right)$ for which

$\langle p \rangle$ has Hilbert fcn ~~$p(D)$ in deg D~~ $p(D+1)$ in deg $D+1$ }

Note: If I is generated in degree D, $I_{D+1} = S_1 I_D$, so $h_I(D+1) = p(D+1)$

iff $\dim_k S_1 I_D = \binom{n+D+1}{n} - p(D+1)$

Macaulay's thm says $h_I(D+1) \leq p(D+1)$

So $\dim_k S_1 I_D \geq \binom{n+D+1}{n} - p(D+1)$.

$$\dim_k S_1 I_D \geq \binom{n+D+1}{n} - p(D+1)$$

So if we want $\dim_k S_1 I_D = \binom{n+D+1}{n} - p(D+1)$

we only need to check $\leq$

This is a determinantal condition:

We form the matrix whose row space

det of submatrix

is a basis for $S_1 I_D$ & set all

minors of size

A has rank $\leq r$ iff all $(r+1)\times(r+1)$ minors are zero

$$\binom{n+D+1}{n} - p(D+1) + 1 \text{ equal to zero.}$$

Example Let $n=1$, $p(t) = 2 = \binom{2+0}{0} + \binom{2-1+0}{0}$

$a_1 = a_2 = 0$

The Gotzmann number is 2, but we will use $D=3$ to illustrate this better.

$\dim_k K[x_0, x_1]_3 = 4$, so are looking for a locus in $Gr(2,4)$ ← coords $P_{mm'}$

m, m' are monomials of deg 3

$4-2$

To make things simpler, we'll work with the affine chart $P_{x_0^3, x_0^2 x_1} \neq 0$.

$$\begin{array}{cccc} x_0^3 & x_0^2x_1 & x_0x_1^2 & x_1^3 \end{array}$$
$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$$

An ideal in this chart has the form $\langle \overset{f}{x_0^3} + a x_0 x_1^2 + b x_1^3,$

$g\ x_0^2 x_1 + c x_0 x_1^2 + d x_1^3 \rangle$

$$\begin{array}{cc} x_0^3 & x_1^3 \end{array}$$
$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \end{pmatrix}$$

The degree 4 part of this ideal "is" the row space of the matrix

$$\begin{array}{c|c} & \begin{array}{ccccc} x_0^4 & x_0^3x_1 & x_0^2x_1^2 & x_0x_1^3 & x_1^4 \end{array} \\ \begin{array}{c} x_0 f \\ x_1 f \\ x_0 g \\ x_1 g \end{array} & \begin{pmatrix} 1 & 0 & a & b & 0 \\ 0 & 1 & 0 & a & b \\ 0 & 1 & c & d & 0 \\ 0 & 0 & 1 & c & d \end{pmatrix} \end{array}$$

We want

$$\dim_k S_1 I_3 \leq \binom{n+D+1}{n} - p(D+1)$$

$$= 5 - 2 = 3$$

So we want all 4×4 minors to vanish

These minors generate the ideal

$$\langle b + cd,\ a + c^2 - d \rangle$$

So the subscheme in this affine chart is $\mathbb{A}^2$.

cf $\mathrm{Hilb}_2(\mathbb{P}^1) = \mathbb{P}^2$, as 2 pts in $\mathbb{P}^1$ is a hypersurface, cut out by the eqtn of degree 2.