TCC Hilbert schemes and moduli spaces

Email to get on mailing list

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If taking for credit, email me your exercise (or discuss alternatives this week)

vevox.app Session id: 193-254-813

Office hours: Friday @ 2pm (in this call)

Today: Universal family, then introduce the Hilbert function

Review A moduli functor $F: \underline{\text{Schemes}} \to \underline{\text{Sets}}$ sends a scheme B to the set of families of objects being parameterized, modulo some equivalence relation (equality, isomorphism)

A scheme X is a fine moduli space for this moduli problem if X represents F ie, $F \simeq h_X = \text{Mor}(-, X)$.

functor of pts

X is unique, if it exists, by Yoneda's lemma.

eg For the Grassmannian we have the moduli functor

$$B \longmapsto \left\{ \begin{array}{l} \text{subsheaves } \mathcal{F} \subseteq \mathcal{O}_B^n \text{ that are} \\ \text{locally free summands of rank } r \end{array} \right\}$$

Restricted to $B = \mathrm{Spec}(R)$

$$R \longmapsto \left\{ \begin{array}{l} \text{submodules } M \subseteq R^{n+1} \text{ that are} \\ \text{locally free direct summands of} \\ \text{rank } r \end{array} \right\}$$

$R = K$ a field $M \subseteq K^n$ is a locally free direct summand of rank r
$\Leftrightarrow$ subspace of dim r.

Universal family

The property that X is a fine moduli space for a functor F is equivalent to the definition of a universal family $\pi : U \to X$ with the property that whenever $\psi : Y \to B$ is a family of the required form ($\psi \in F(B)$) there is a unique morphism $\phi : B \to X$ st

$$\begin{tikzcd} Y = U \times_X B \arrow[r] \arrow[d, "\psi"'] & U \arrow[d, "\pi"] \\ B \arrow[r, "\exists!"] & X \end{tikzcd}$$

pullback diagram/
fibre product

set theoretic:
$\{(u,b) \in U \times B : \pi(u) = \phi(b)\}$

$$\begin{tikzcd} Y = U \times_X B \arrow[r] \arrow[d] & U \arrow[d, "\pi"] \\ B \arrow[r, "\exists !"'] & X \end{tikzcd}$$

If X represents F (so $F \simeq \mathrm{Mor}(-,X)$) set $\pi: U \to X$ to be the element of $F(X)$ corresponding to $id: X \to X \in \mathrm{Hom}(X,X)$

(Ex: Check that this implies the pull back property)

If $\pi: U \to X$ is a universal family, for each B we get a function $\alpha_B: F(B) \to \mathrm{Hom}(B,X)$

This is the data of a natural trans. (Ex!) and is an isomorphism, so X represents F.

eg The universal family of $\mathbb{P}^2$ is

$$U = \{ [x_0 : x_1 : x_2], (y_0, y_1, y_2) \in \mathbb{P}^2 \times \mathbb{A}^3 : \operatorname{rk}\begin{pmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \end{pmatrix} = 1 \}$$

2x2 minors are zero

The map $\pi : U \to \mathbb{P}^2$ is projection onto the first factor and the fibre over a pt $[x] \in \mathbb{P}^2$ is the line through the origin spanned by $x \in \mathbb{A}^3$.

$\mathbb{P}^2$: $\quad R \mapsto \left\{ \begin{array}{l} \text{submodules of } R^3 \\ \text{locally free direct summands} \\ \text{of rank } 2 \end{array} \right\}$

(subs vs quotients)

$$R^3 = M \oplus R\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$M = R^3 / R\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

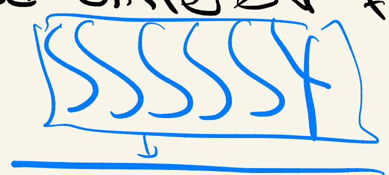
The Hilbert functor

First attempt: $\mathrm{Hilb}(\mathbb{P}^n) : \underline{\text{Schemes}} \longrightarrow \underline{\text{Sets}}$

is given by $\mathrm{Hilb}(\mathbb{P}^n)(B) = \left\{ \begin{array}{l} \text{subschemes} \\ Z \subseteq \mathbb{P}^n_B = \mathbb{P}^n \times B \\ \text{that are flat} \\ \text{over } B \end{array} \right\}$

Here "flat" is niceness property for bundles in algebraic geometry that guarantees that the fibres of the morphism $Z \to B$ are not too different from each other

(weaker than fibre bundles in topology)



Commutative alg

An R-module M is flat if $- \otimes M$ is an exact functor.

If $0 \to A \to B \to C \to 0$ is an exact seq of R-modules, we always have

$$A \otimes M \to B \otimes M \to C \otimes M \to 0$$

"tensor is right exact".

The content is requiring $0 \to A \otimes M \to B \otimes M$ to be exact as well.

eg If $M = R^n$ M is flat

$\mathbb{Z}/2\mathbb{Z}$ is not a flat $\mathbb{Z}$-module:

$$0 \to \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

Geometric defn:

If $\phi^\#: A \to B$ is a ring homomorphism, making B into an A-module, then $\phi: \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is flat if B is a flat A-module.

In general $\phi: X \to Y$ is flat if the stalk $\mathcal{O}_{X,x}$ of X at a pt x is flat as an $\mathcal{O}_{Y,y}$-module where $y = \phi(x)$.

Useful fact: If A is a PID eg $k[t]$ $\cancel{\mathbb{Z}}$ then $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is flat if & only if B is a torsion free A-module. Ref: Eisenbud CommAlg Cor 6.3

Issue: this Hilbert functor is too "big"
- the scheme representing it will have ∞-many components

We make it more manageable by also imposing restrictions on the Hilbert polynomial of fibres.

Notation For now, work over a field K.

will apply when K is the residue field of a fibre.

$S = K[x_0, \ldots, x_n]$

graded: $\deg(x_i) = 1$ $\forall i$, so $S = \bigoplus_{d \geq 0} S_d$

An ideal $I \subseteq S$ is homogeneous if it is generated by homogeneous elements

Then $S/I \cong \bigoplus_{d \geq 0} (S/I)_d \cong \bigoplus_{d \geq 0} S_d/I_d$

Key fact A subscheme $Z \subseteq \mathbb{P}^n_K$ is determined by a homogeneous ideal $I \subseteq K[x_0, \ldots, x_n] = S$

$$\mathcal{O}_Z \cong \mathcal{O}_{\mathbb{P}^n}/\mathcal{I}_Z \qquad \mathcal{I}_Z = \tilde{I} \leftarrow \text{sheafification}$$

On affine charts: $I_i = (I S[x_i^{-1}])_0 \subseteq S[x_i^{-1}]_0$

Then the intersection of Z with the affine chart $D(x_i) = \{x_i \neq 0\}$ is $\quad S[x_i^{-1}]_0 = K[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$

$$\operatorname{Spec}\left(K\left[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}\right] \Big/ I_i \right).$$

The correspondence between subschemes of $\mathbb{P}^n$ & homogeneous ideals in $K[x_0, x_n]$ is not 1-1; two different ideals can have the same sheafification.

Ideals $I, J \subseteq K[x_0, x_n]$ correspond to the same subscheme if their saturations agree:

$$(I : \langle x_0, , x_n \rangle^\infty) = \{ f \in S : \exists k > 0 \text{ with } fm \in I \text{ for all } m \in \langle x_0, x_n \rangle^k \}$$

$$(I : \langle x_0, , x_n \rangle^\infty) = (J : \langle x_0, x_n \rangle^\infty).$$

$(I : \langle x_0 \dots x_n \rangle^\infty) = \{ f \in S : \exists k > 0 \text{ with } fm \in I \ \forall m \in \langle x_0 \dots x_n \rangle^k \}$

eg $I = \langle x_1 \rangle \quad J = \langle x_1^2, x_0 x_1 \rangle = \langle x_1 \rangle \cap \langle x_0, x_1^2 \rangle$ in $K[x_0, x_1]$. I & J both have saturation $\langle x_1 \rangle$. The associated subscheme is $[1:0]$ with the reduced scheme structure.

In general, I, J have the same saturation iff $I_d = J_d$ for $d \gg 0$.

Defn The Hilbert function of a homogeneous ideal $I \subseteq K[x_0, \ldots, x_n]$ is the function $h_I : \mathbb{N} \to \mathbb{N}$ given by $h_I(d) = \dim_K (S/I)_d$.

eg $I = \langle 0 \rangle$

$$h_I(d) = \dim_K S_d$$

$$= \binom{n+d}{d} = \binom{n+d}{n} \leftarrow \text{Ex!}$$

$$= \frac{1}{n!}(n+d)(n+d-1)\cdots(n+1)$$

↑ polynomial in d of degree n.

eg $I = \langle f \rangle$ where f is homogeneous of degree m.

$$h_I(d) = \begin{cases} \dim_k S_d & d < m \\ \dim_k S_d - \dim_k S_{d-m} & d \geq m \end{cases}$$

$$= \begin{cases} \binom{n+d}{n} & d < m \\ \binom{n+d}{n} - \binom{n-m+d}{n} & d \geq m \end{cases}$$

Note that $h_I(d)$ is a polynomial in d of degree $n-1$ for $d \gg 0$

$d \geq m$

For a homogeneous ideal $I \subseteq S$, $h_I(d)$ agrees with a polynomial $p_I \in \mathbb{Q}[t]$ for $d \gg 0$. This polynomial p_I is called the Hilbert polynomial of I (of S/I)

When Z is a subscheme of $\mathbb{P}^n$ with ideal I, $\dim(Z) = \deg(p_I)$. The degree of Z is $\dim(Z)!$ times the leading coeff.

Geometrically,

$$P_I(d) = \chi\left(\mathcal{O}_Z(d)\right) = \sum_{i=0}^{\dim Z} (-1)^i \dim_k H^i(\mathbb{P}^n, \mathcal{O}_Z(d))$$

Serre vanishing

$$= \dim_k H^0(\mathbb{P}^n, \mathcal{O}_Z(d))$$

for $d \gg 0$

eg When Z is a hypersurface of degree m

$$P_I(t) = \binom{n+t}{n} - \binom{n-m+t}{n}.$$

eg When Z is a line in $\mathbb{P}^n$, $P_I(t) = t+1$.

eg When Z is a smooth curve of degree d & genus g $\quad P_I(t) = dt + 1 - g$

If Z is embedded into $\mathbb{P}^n$ by the complete linear series $L(D)$, this is Riemann Roch.

RR: $\quad \ell(D) - \ell(K-D) = \deg(D) + 1 - g$

$$\ell(D) - \ell(K-D) = \deg(D) + 1 - g$$

Apply this to tD, using that

$$h_I(t) = \ell(tD)$$

Since $\ell(K-tD) = 0$ for $t >> 0$

$$\begin{aligned} h_I(t) &= \deg(tD) + 1 - g \\ &= t\deg(D) + 1 - g \\ &= dt + 1 - g \end{aligned}$$

New Hilbert Functor

$\mathrm{Hilb}_P(\mathbb{P}^n) : \underline{\text{Schemes}} \longrightarrow \underline{\text{Sets}}$

given by

$\mathrm{Hilb}_P(\mathbb{P}^n)(B) = \{$ subschemes $Z \subseteq \mathbb{P}^n_B$ st $Z \to B$ is a flat family with every fibre having Hilbert polynomial P $\}$

We will show that this is representable by a projective scheme $\mathrm{Hilb}_P(\mathbb{P}^n)$. The original Hilbert functor is then represented by the disjoint union over all P ← countable union eg When $P(t) = r$ is constant, $\mathrm{Hilb}_P(\mathbb{P}^n)$ is the Hilbert scheme of r pts in $\mathbb{P}^n$. This has an irreducible component parameterizing subschemes consisting of r distinct pts in $\mathbb{P}^n$ (and their degens) but it can have other components

eg When $P(t) = t+1$, Z is a line in $\mathbb{P}^n$ and $\mathrm{Hilb}_P(\mathbb{P}^n) = \mathrm{Gr}(2, n+1)$

eg When $P(t) = 2t+1$, $\mathrm{Hilb}_P(\mathbb{P}^2)$ parameterizes conics in $\mathbb{P}^2$

These can be described by an equation

$$\star \quad a x_0^2 + b x_0 x_1 + c x_0 x_2 + d x_1^2 + e x_1 x_2 + f x_2^2 = 0$$

for $[a:b:c:d:e:f] \in \mathbb{P}^5$

$$\begin{tikzcd} K[x_0, x_1, x_2] \arrow[r] \arrow[d, "f"] & U \arrow[d, phantom, "\cup"] \\ \mathrm{Spec}(K) \arrow[r] & \mathbb{P}^5 \end{tikzcd}$$

so $\mathrm{Hilb}_{2t+1}(\mathbb{P}^2) = \mathbb{P}^5$

The eqn $\star$ defines the universal family of $\mathrm{Hilb}_{2t+1}(\mathbb{P}^2)$ in $\mathbb{P}^2 \times \mathbb{P}^5$

In general, when $P(t) = \binom{n+t}{n} - \binom{n-m+t}{n}$ the Hilbert scheme $\mathrm{Hilb}_P(\mathbb{P}^n)$ parameterizes hypersurfaces in $\mathbb{P}^n$ of degree m, so $\mathrm{Hilb}_P(\mathbb{P}^n) \simeq \mathbb{P}^{\binom{t+r}{r}-1}$

Implicit exercise: If Z has Hilbert poly P, Z is a hypersurface of degree m.

eg If $P(t) = \binom{t+r}{r}$ then if Z has Hilbert polynomial P, Z is a subspace of $\mathbb{P}^n$ of dim r (ex!) so $\mathrm{Hilb}_P(\mathbb{P}^n) = Gr(r+1, n+1)$

If P is the Hilbert polynomial of all subschemes with a given property

(eg dim $n-1$, deg r

~~or~~ degen of r distinct pts) it does not always follow that every subscheme with Hilbert polynomial P has this property

eg The "twisted cubic" (image of the Veronese embedding of $\mathbb{P}^1$ into $\mathbb{P}^3$) has Hilbert $3T+1$.

However $\mathrm{Hilb}_{3T+1}(\mathbb{P}^3)$ has 2 comp.

one generically parameterizes twisted cubics
& the other generically parameterizes
a plane cubic plus a pt.

Ref: Piene - Schlessinger

This happens routinely with moduli spaces

Next: Construct $\mathrm{Hilb}_p(\mathbb{P}^n)$ as a subscheme of a Grassmannian.