TCC Hilbert schemes and moduli spaces

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Today Pathologies of Hilbert schemes

The smoothable component of $\mathrm{Hilb}^N(\mathbb{A}^d)$

We first show that the closure of the locus of N distinct reduced pts in $\mathbb{A}^d$ is an irreducible component of $\mathrm{Hilb}^N(\mathbb{A}^d)$ for any d.

dim Nd

locus is irreducible, so lives in one component

It suffices to find a point in this locus whose tangent space has dimension at most Nd. That pt will then be smooth & the dimension of the component is Nd.

Consider $I = \langle f, x_2, \ldots, x_d \rangle \subseteq K[x_1, \ldots x_d] = S$ where f is a polynomial in x_1 of degree N. Then $\dim_K(S/I) = N$, so $[I] \in \mathrm{Hilb}^N(\mathbb{A}^d)$.

Recall The tangent space to $[I]$ is $\mathrm{Hom}_S(I, S/I)$.

For $\varphi : I \to S/I$, φ is determined by $\varphi(f), \varphi(x_2), \ldots, \varphi(x_d)$. Since $\dim_K S/I = N$, the space of choices for these has dimension dN, so $\dim_K \mathrm{Hom}_S(I, S/I) \leq dN$.

Since dN is our lower bound for the dimension of any component containing $[I]$, so $\dim_k \mathrm{Hom}_S(I, S/I) = dN$, and this is a smooth point.

The component that is the closure of the locus of N reduced points is called the **smoothable component**.

It has an explicit description as a blow-up of $(\mathbb{A}^d)^N / S_N$ ← Chow variety

$\mathrm{Spec}(k[(\mathbb{A}^d)^N]^{S_N})$

The Hilbert scheme of pts in $\mathbb{A}^3$

① $\mathrm{Hilb}^N(\mathbb{A}^3)$ is singular for $N \geq 4$.

eg $N=4$

$m = \langle x^2, xy, xz, y^2, yz, z^2 \rangle = \langle x,y,z \rangle^2$

$1, x, y, z$ $\subseteq k[x,y,z]$

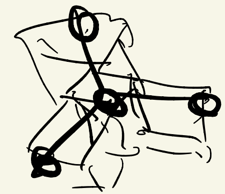
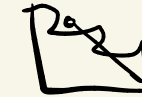
$\varphi \in \mathrm{Hom}(m, S/m)$

$\varphi(x^2) = \cancel{a} + bx + cy + dz$ for some a,b,c,d.

$\varphi(xy) = e + fx + gy + hz$ - - - -

$\underbrace{y\varphi(x^2)}_{ay} = \underbrace{x\varphi(xy)}_{ex}$ in $S/m \Rightarrow a = e = 0$

no conditions on other terms!

$m = \langle x^2, xy, xz, y^2, yz, z^2 \rangle$  

$$\dim_k \mathrm{Hom}_S\left(m, S/m\right)$$

$$= 18 = \underset{\text{generators}}{6} \times \underset{\text{parameters}}{3}$$

coeffs of x, y, z in φ(gen)

$$> 12 = \underset{N}{4} \times \underset{d}{3}$$

(since m lies on the smoothable component

So $[m]$ is not a smooth pt of $\mathrm{Hilb}^4(\mathbb{A}^3)$

② We next show $\mathrm{Hilb}^N(\mathbb{A}^3)$ is reducible for $N \gg 0$.

dim $> 3N$ = dim smooth comp

We do this by showing a large Grassmannian embeds into $\mathrm{Hilb}^N(\mathbb{A}^3)$

Construction: Set $S = K[x,y,z]$.

Fix a degree r & $0 < s < \dim_K S_r = \binom{r+2}{2}$

Set $N = \sum_{i=0}^{r-1} \dim_K S_i + s = \binom{r-1+3}{3} + s$

$= \binom{r+2}{3} + s.$

For every subspace $L \subseteq S_r$ of dim $\binom{r+2}{2} - s$.

The ideal $I_L = \langle L \rangle + S_{\geq r+1}$ has $\dim_K S/I_L = N$

$I_L = \langle L \rangle + S_{\geq r+1}$ for $L \in Gr\left(\binom{r+2}{2} - s, S_r\right)$.

This gives a flat over $Gr\left(\binom{r+2}{2} - s, \binom{r+2}{2}\right)$,
so an embedding of $Gr\left(\binom{r+2}{2} - s, \binom{r+2}{2}\right)$
into $Hilb^N(\mathbb{A}^3)$

To show $Hilb^N(\mathbb{A}^3)$ is reducible, it suffices to choose r, s so that the dimension of the Grassmannian is greater than the dimension of the smoothable component.

ie $\quad s\left(\binom{r+2}{2} - s\right) > 3\binom{r+2}{3} + 3s \leftarrow 3N$

want: $s\left(\binom{r+2}{2}-s\right) > 3\binom{r+2}{3}+3s$.

Choose $r \equiv 3 \mod 4$ so $\binom{r+2}{2} = \frac{1}{2}(r+2)(r+1)$ is even.

Set $s = \frac{1}{2}\binom{r+2}{2}$.

Then dim Grassmannian

$= s\left(\binom{r+2}{2}-s\right) = \frac{1}{4}\binom{r+2}{2}^2$ ← poly in r of deg 4

dim smoothable comp.

$= 3N = 3\binom{r+2}{3} + \frac{3}{2}\binom{r+2}{2}$ ← poly in r of deg 3

For $r \gg 0$ the dimension of the Grassmannian is large so $\mathrm{Hilb}^N(\mathbb{A}^3)$ is reducible

Iarrobino Version of this argument shows $\mathrm{Hilb}^N(\mathbb{A}^3)$ is reducible for $N \geq 78$

We know $\mathrm{Hilb}^N(\mathbb{A}^3)$ is irreducible

for $N \leq 11$, + 2017

$12 \leq N \leq 77$????

For $d \geq 4$ $\mathrm{Hilb}^N(\mathbb{A}^d)$ is reducible for $N \geq 8$ & irreducible for $N < 8$ Poonen

CEVV 2008

For smoothness for arbitrary $\mathrm{Hilb}_P(\mathbb{P}^n)$ see recent work of Skjelnes-Smith 2020 characterizing P with $\mathrm{Hilb}_P(\mathbb{P}^N)$ smooth.

Murphy's Law for Hilbert schemes

"There is no geometric possibility so horrible that cannot be found generically on some component of some Hilbert scheme" [Harris-Morrison]

Philosophy attributed to Mumford who showed $\mathrm{Hilb}^{dt+1-g}_{14t-23}(\mathbb{P}^3)$ has a component that is everywhere nonreduced, even though the curves on this component are (generically) smooth & irreducible, of degree 14 & genus 24. They lie on a smooth cubic surface S & live in a prescribed linear equiv class

Made formal by Vakil who showed every singularity type appears on some Hilbert scheme

A morphism $\varphi: X \to Y$ of schemes of finite type over K is smooth if it is flat & every fibre is geometrically regular (think: still nonsingular when pass to alg closure)

$\varphi: (X,p) \to (Y,q)$ is a smooth morphism of pointed schemes if $p \in X$, $q \in Y$, φ is a smooth morphism with $\varphi(p) = q$

Defn A **singularity type** is an equivalence class of pointed schemes defined by setting $(X,p) \sim (Y,q)$ if $\varphi : (X,p) \to (Y,q)$ is a smooth morphism.

eg The projection $X \times \mathbb{A}^n \to X$ is smooth so $(X \times \mathbb{A}^n, (p,0)) \sim (X,p)$ for any pt $p \in X$.

Defn We say **Murphy's Law holds** for a moduli space M if every singularity type of pointed schemes appears on M. This means that there is a pt $q \in M$ with completed local ring $\hat{\mathcal{O}}_{M,q}$ isomorphic to $\hat{\mathcal{O}}_{X,p}$ for some representative (X,p) of the singularity type.

Theorem Vakil

The Hilbert scheme $\mathrm{Hilb}(\mathbb{P}^n)$ satisfies Murphy's law (for large n).

In particular, this holds for

$$\mathrm{Hilb}_{\text{surfaces}}(\mathbb{P}^4) \leftarrow \coprod_{\deg P = 2} \mathrm{Hilb}_P(\mathbb{P}^4)$$

Q/ Why singularity types?

Consider $(X, p) = \left(\operatorname{Spec} \frac{k[x]}{(x^2)}, \langle x \rangle\right)$ isolated double pt.

Suppose $\frac{k[x]}{(x^2)}$ is the local ring of a pt

$[X]$ on $\mathrm{Hilb}_P(\mathbb{P}^n)$. Then $[X]$ is an zero-dimensional component

$PGL(n+1)$ acts on $\mathrm{Hilb}_P(\mathbb{P}^n)$

so $[X]$ must be fixed by this action.

But this means $X = \mathbb{P}^n$ or $X = \emptyset$

(the saturated ideal I of X is Borel-fixed so if $I \neq 0$, I contains a power of x_0. & symmetric so contains a power of x_i for all i, but then $I \supseteq \langle x_0, \ldots x_n \rangle^m$, so $X = \emptyset$.)

eg $P = \binom{t+n}{n}$. So $\mathrm{Hilb}_P(\mathbb{P}^n)$ is one reduced pt.

So (X,p) does not appear in any $\mathrm{Hilb}_P(\mathbb{P}^n)$.

However we can still look for something in the equivalence class

$$\text{Leg}\quad \operatorname{Spec}\left(\frac{k[x_1, \ldots, x_n]}{(x_1^2)}, (0, \ldots, 0)\right)$$

for $n > 1$.

Consequences of Murphy's Law

eg Non-reduced pts exist.

$(X,p) = \left(\operatorname{Spec} \frac{k[x]}{x^2}, 0\right)$. Any $(Y,q) \mapsto (X,p)$

is ~~not~~ not reduced at q

eg There are components that only exist in characteristic p.

ie
$$\begin{array}{c} \mathrm{Hilb}_p(\mathbb{P}^n_{\mathbb{Z}}) \\ \downarrow \\ \operatorname{Spec}(\mathbb{Z}) \end{array}$$

← some components live only over $\langle p \rangle$.

$(X,p) = (\operatorname{Spec}(\mathbb{Z}/p\mathbb{Z}), 0)$.

If $(Y,q) \sim (X,q)$ then $p\mathcal{O}_{Y,q} = 0$

So this only occurs in characteristic p

Theorem [Jelisiejew]

might not be minimal

Murphy's law holds for $\operatorname{Hilb}_{pts}(\mathbb{A}^{16})$

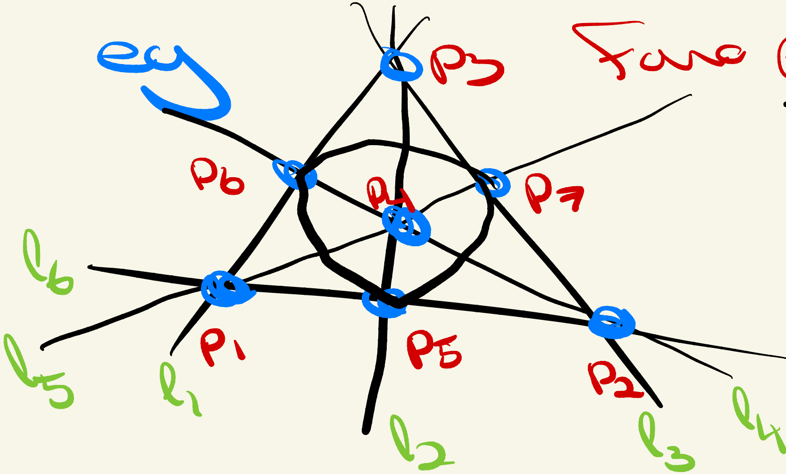
Key idea in the proof

Reduce to another moduli space where Murphy's law holds: the moduli space of point-line incidences in $\mathbb{P}^2$

(secretly realization spaces of matroids)

Defn An incidence scheme of pts & lines in $\mathbb{P}^2$ is a locally closed subscheme of $(\mathbb{P}^2)^m \times (\mathbb{P}^{2\vee})^n = \{(p_1, \ldots, p_m) \times (l_1, \ldots, l_n)\}$ parameterizing $m \geq 4$ distinct pts & n distinct lines with prescribed incidences & nonincidences (p_i lies on line l_j or p_i does not lie on line l_j).

Normalise: $p_1 = [1:0:0]$, $p_2 = [0:1:0]$, $p_3 = [0:0:1]$, $[1:1:1]$. We require any two lines to intersect in a marked pt & any two lines contain ≥ 3 marked pts



Fano plane $\mathbb{P}^2_{\mathbb{F}_2}$

The diagram encodes the required incidences/nonincidences

$$\left(\begin{array}{ccccccc} \left[\begin{matrix} 1 & 0 & 0 & 1 & \cancel{a_1} & \cancel{b_1} & c_1 \\ 0 & 1 & 0 & 1 & \cancel{a_2} & b_2 & \cancel{c_2} \\ 0 & 0 & 1 & 1 & a_3 & \cancel{b_3} & \cancel{c_3} \end{matrix}\right] \\ p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5 \quad p_6 \quad p_7 \end{array}, \begin{array}{c} \left[\begin{matrix} d_1 & \cancel{e_1} & \cancel{f_1} & \cancel{g_1} & h_1 & i_1 & j_1 \\ \cancel{d_2} & \cancel{e_2} & f_2 & g_2 & \cancel{h_2} & i_2 & j_2 \\ d_3 & e_3 & f_3 & \cancel{g_3} & \cancel{h_3} & \cancel{i_3} & j_3 \end{matrix}\right] \\ l_1 \quad l_2 \quad l_3 \quad l_4 \quad l_5 \quad l_6 \quad l_7 \end{array} \right)$$

l_1 contains p_1, p_3, p_6 $\qquad$ l_2: p_3, p_4, p_5

$(1,0,0)\cdot(d_1, d_2, d_3) = 0$ $\qquad$ l_3: p_2, p_3, p_7

$d_1 = 0 = d_3 \quad d_2 = 1$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & j_1 \\ 1 & -1 & 0 & 0 & 1 & 0 & j_2 \\ 0 & 0 & 0 & -1 & -1 & 1 & j_3 \end{bmatrix}$$

P_4, P_5, P_6 lie on l_7

$$j_1 + j_2 = j_1 + j_3 = j_2 + j_3 = 0$$

Nonzero soln iff $\mathrm{char}(K) = 2$

So the incidence scheme is a reduced pt in char 2 & empty o/w

Theorem [Mnëv - Sturmfels universality]

The disjoint union of all incidence schemes satisfies Murphy's law.

see also Lafforgue

Idea of the pf Reduce to singularity is $\operatorname{Spec}\left(\frac{k[x_1, \ldots, x_n]}{\langle f_1, \ldots, f_r \rangle}\right)$ ← rewrite in terms of atomic relations