TCC Hilbert Schemes and moduli spaces

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Today: Smoothness

1) Tangent space to $\mathrm{Hilb}_P(\mathbb{P}^n)$
2) Smoothness of $\mathrm{Hilb}^N(\mathbb{A}^2)$

Tangent space to $\mathrm{Hilb}_P(\mathbb{P}^n)$

Recall that the Zariski tangent space to a scheme X at a k-rational pt p is $\mathrm{Hom}(m/m^2, k)$

where m is the maximal ideal of $\mathcal{O}_{X,p}$ & $k = \kappa(p) = \mathcal{O}_{X,p}/m$

residue field

* (might be char / perfect assumptions)

Lemma A $K[\varepsilon]/\varepsilon^2$-valued pt of X $\leftrightarrow \mathrm{Hom}(\mathrm{Spec}(K[\varepsilon]/\varepsilon^2), X)$

is a K-rational pt p of X together with an element of the Zariski tangent space to X at p

Eisenbud Harris Chapter VI

Proof.

dual numbers

The K-algebra homomomorphism $K[\varepsilon]/\varepsilon^2 \longrightarrow K$ induces a morphism

$\varepsilon \longmapsto 0$

$\mathrm{Spec}(K) \longrightarrow \mathrm{Spec}(K[\varepsilon]/\varepsilon^2)$, so a $K[\varepsilon]/\varepsilon^2$-valued pt of X induces a K-rational pt of X.

We also get a local homomorphism

$$\mathcal{O}_{X,p} \longrightarrow K[\varepsilon]/\varepsilon^2$$

which induces

$$m \longmapsto \langle \varepsilon \rangle$$

$$m/m^2 \longrightarrow \langle \varepsilon \rangle \cong K$$

↑ in Zariski tangent space

Conversely given $p \in X$ + $t: m_p/m_p^2 \longrightarrow K \cong \langle \varepsilon \rangle$

note that $\mathcal{O}_{X,p}/m_p^2 \cong \underbrace{\mathcal{O}_{X,p}/m}_{\cong K} \oplus m_p/m_p^2$

Define $\psi: \mathcal{O}_{X,p}/m_p^2 \longrightarrow K[\varepsilon]/(\varepsilon^2)$ by identity on $\mathcal{O}_{X,p}/m$ and t on m_p/m_p^2

We then get a ring homomorphism $\varphi: \mathcal{O}_{X,p} \to K[\varepsilon]/\varepsilon^2$ and so a morphism $\operatorname{Spec}(K[\varepsilon]/\varepsilon^2) \to X$ $\square$

Consequence The tangent space to $\operatorname{Hilb}_p(\mathbb{P}^n)$ at a pt $[X]$ is an element of $\operatorname{Hom}(\operatorname{Spec}(K[\varepsilon]/\varepsilon^2), \operatorname{Hilb}_p(\mathbb{P}^n))$ that maps $\operatorname{Spec}(K)$ to $[X]$

Tangent space at $[X] \longleftrightarrow \mathrm{Hom}(\mathrm{Spec}(k[\varepsilon]/\varepsilon^2), \mathrm{Hilb}_p(\mathbb{P}^n))$

This set is in natural correspondence with the set of flat families

$$\begin{array}{c} \mathcal{X} \subseteq \mathbb{P}^n_{k[\varepsilon]/\varepsilon^2} \\ \downarrow \\ \mathrm{Spec}(k[\varepsilon]/\varepsilon^2) \end{array}$$

where the fibre over $\langle \varepsilon \rangle$ is X

The space of such flat families is called the space of first order deformations of X in $\mathbb{P}^n_k$

Definition The **normal sheaf** to a closed subscheme X of a scheme Y is the sheaf

$$\mathcal{N}_{X/Y} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}, \mathcal{O}_X)$$

where $\mathcal{I}$ is the ideal sheaf of X in Y.

eg if X is the subscheme of $\mathbb{A}^n$ determined by an ideal $I \subseteq S := k[x_1, \dots, x_n]$ the normal sheaf is the sheafification of

$$\mathrm{Hom}_S(I, S/I)$$

$\mathrm{Hom}_S(I, S/I)$

Eisenbud-Harris Thm VI-29

Theorem The space of first order deformations of a closed subscheme X of a scheme Y is the space of global sections of the normal sheaf $\mathcal{N}_{X/Y}$.

Idea when $Y = \mathbb{A}^2$

a family $\mathcal{X} \subseteq \mathbb{A}^2 \times \mathrm{Spec}\left(k[\varepsilon]/\varepsilon^2\right)$

over $\mathrm{Spec}\left(k[\varepsilon]/\varepsilon^2\right)$ has $\mathcal{X} = \mathrm{Spec}\left(\dfrac{k[x,y,\varepsilon]}{(\varepsilon^2) + J}\right)$

where $J = \langle f_1 + \varepsilon g_1, \ldots, f_s + \varepsilon g_s \rangle$

where f_1, f_s generate the ideal I of X; $f_i, g_i \in k[x,y]$

$J = \langle f_1+\varepsilon g_1, \dots, f_s+\varepsilon g_s\rangle$, $I = \langle f_1, \dots, f_s\rangle$

Key idea: $\varphi : I \longrightarrow k[x,y]/I \in \mathrm{Hom}(I, S/I)$

given by $\varphi(f_i) = g_i$ exists if and only if

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \mathrm{Spec}(k[\varepsilon]/\varepsilon^2) \end{array} \text{ is flat.}$$

sections of normal bundle

Smoothness of $\mathrm{Hilb}^N(\mathbb{A}^2)$

The Hilbert scheme of N pts in $\mathbb{A}^2$ is the locus in $\mathrm{Hilb}_p(\mathbb{P}^2)$ for $p(t)=N$ of subschemes supported in $\mathbb{A}^2 \subseteq \mathbb{P}^2$. We now show this is smooth.

Key special case of Fogarty's result that Hilbert schemes of pts on smooth surfaces are smooth & irreducible.

We follow the approach of Haiman.

See also Miller-Sturmfels

Step 1 The singular locus (if nonempty) contains a monomial ideal

(true for all $\mathrm{Hilb}_P(\mathbb{P}^n)$)

This follows from the Gröbner degeneration flat family

$$\mathrm{Spec}\left(\frac{k[x,y,t]}{I_t}\right) \to \mathbb{A}^1_t$$

$w \in \mathbb{N}^2$

$f \mapsto f\left(\frac{x}{t^{w_1}}, \frac{y}{t^{w_2}}\right)$

where $I_t = \langle \check{f} : f \in I \rangle$

$\mathrm{Hom}(I, S/I)$

This determines a morphism $\varphi : \mathbb{A}^1 \to \mathrm{Hilb}^N(\mathbb{A}^2)$ with $\varphi(1) = [I]$. If (I) is a singular pt of $\mathrm{Hilb}^N(\mathbb{A}^2)$, so is $[tI]$ for $t \neq 0$

fibre over $t \in \mathbb{A}^1$: $\varphi(t)$

The singular locus of any scheme is closed, so the special fiber $\varphi(0)$ must also be a singular pt.

This is $[\mathrm{in}_w(I)]$.

initial ideal wrt w

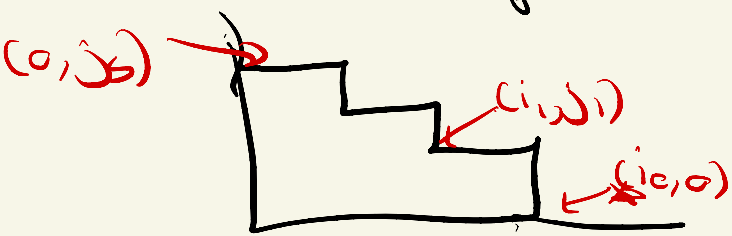
For general $w \in \mathbb{N}^2$ this is a monomial ideal, so if $\mathrm{Hilb}^N(\mathbb{A}^2)$ has a singular pt, there is a monomial ideal that is singular.

Back at 11:03

Step 2 The dimension of $\mathrm{Hilb}^N(\mathbb{A}^2)$ at a monomial ideal is at least $2N$

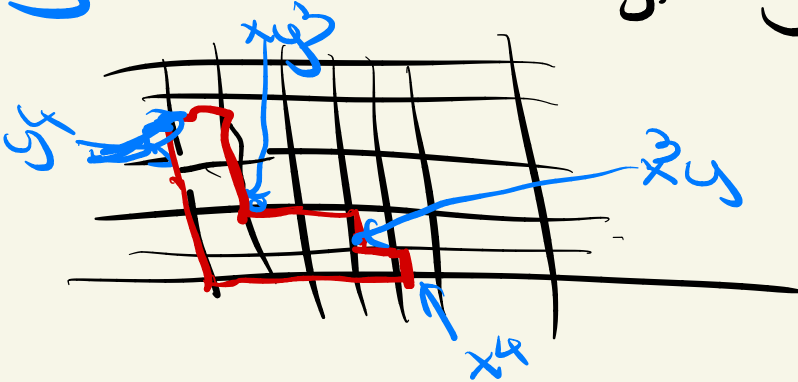
A monomial ideal $\mathfrak{m} \subseteq S = k[x,y]$ is a pt in $\mathrm{Hilb}^N(\mathbb{A}^2)$ if $\dim_k S/I = N$

We can represent $\mathfrak{m}$ by its staircase diagram



$$\mathfrak{m} = \langle x^{i_0}, x^{i_1}y^{j_1}, \ldots, x^{i_{s-1}}y^{j_{s-1}}, y^{j_s} \rangle$$

eg $m = \langle x^4, x^3y, xy^2, y^4 \rangle$



$N = 9$

The number of boxes under the staircase is N
(since monomials not in m form a basis for S/m)
This is the Ferrers shape / Young diagram
of a partition of N (see oeis.org)

eg $9 = 4 + 3 + 1 + 1$

$\rightsquigarrow$ The number of monomial ideals in $\mathrm{Hilb}^N(\mathbb{A}^2)$
is the number of partitions of N.

Fix a monomial ideal m with $\dim_K(S/m) = N$

We'll assume $\operatorname{char}(K) = 0$

Consider $\{(i,j) \in \mathbb{N}^2 : x^i y^j \notin m\}$ as a collection of N pts in $\mathbb{A}^2$.

Let I be the ideal of polynomials vanishing on these pts (so $\dim_K S/I = N$)

For each generator $x^i y^j$ of m $\quad x^2y^2$

construct $f = \prod_{k=0}^{i-1}(x-k)\prod_{l=0}^{j-1}(y-l) \quad x(x-1)y(y-1)$

distraction of monomial

Note that $f \in I$.

$$f = \prod_{k=0}^{i-1}(x-k)\prod_{l=0}^{j-1}(y-l)$$

For any $w \in \mathbb{N}^2_{\geq 0}$ $\mathrm{in}_w(f) = x^i y^j$

So $M \subseteq \mathrm{in}_w(I)$. Since $\dim_k(S/M) = \dim_k(S/I) = N$

preserved by passing to an initial ideal.

we must have $M = \mathrm{in}_w(I)$

Thus M is the limit of a family of N distinct pts

So $[M]$ lies on the same irreducible component as the locus of N distinct pts..

The locus of N distinct reduced pts

is $\left((\mathbb{A}^2)^N \setminus \left\{ \begin{matrix} \text{diagonal} \\ (x_i, y_i) = (x_j, y_j) \end{matrix} \right\} \right) / S_N$

which has dimension $2N$,

so the irreducible component has

dimension $\geq 2N$.

(argument generalizes to $\mathrm{Hilb}^N(\mathbb{A}^d)$ for any d) $\Rightarrow$ radius of $\mathrm{Hilb}^N(\mathbb{A}^d)$ is ≤ 1. (cf Reeves Radius of Hilbert scheme)

Step 3 The target space of a monomial ideal is $2N$-dimensional

We want to compute $\dim_k \mathrm{Hom}(m, S/m)$

An element $\varphi \in \mathrm{Hom}(m, S/m)$ is determined by choosing where φ sends the generators of m, subject to the requirement that these choices are compatible.

we need

eg $m = \langle x^4, x^3y, xy^3, y^4 \rangle$ $\varphi(x^4y) = y\varphi(x^4)$

$= x\varphi(x^3y)$

Defn The first syzygy module of a graded module P is the module of relations among the minimal generators of P. If $p_1, \ldots, p_m$ generate P as a S-module,

$$\mathrm{Syz}(P) = \{ (r_1, \ldots, r_m) \in S^m : \sum r_i p_i = 0 \}$$

$$0 \leftarrow P \leftarrow S^m \leftarrow \mathrm{Syz}(P) \leftarrow 0$$

eg For $m = \langle x^4, x^3y, xy^2, y^4 \rangle$

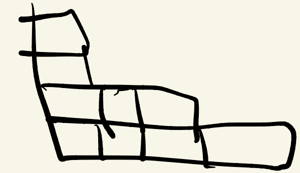
$(-y, x, 0, 0)$ is a syzygy.

Consider $M = \langle x^{i_k} y^{j_k} : 0 \leq k \leq s \rangle$

For $(f_0, \dots, f_s)$ to define an element $\varphi \in \mathrm{Hom}(M, S/M)$ by $\varphi(x^{i_k} y^{j_k}) = f_k$ we need $\sum r_i f_i = 0 \in S/M$ for all $(r_0, \dots, r_s) \in \mathrm{Syz}(M)$

Lemma The syzygy module of a monomial ideal $M \subseteq K[x,y]$ is generated by "adjacent syzygies".

eg $M = \langle \overset{0}{x^4}, \overset{1}{x^3y}, \overset{2}{xy^2}, \overset{3}{y^4} \rangle$

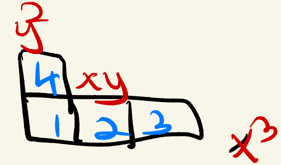


$$\mathrm{Syz}(M) = S\{(-y, x, 0, 0), (0, -y, x^2, 0), (0, 0, -y^2, x)\}$$

Write $m_1, \dots, m_N$ for the monomials not in M.

We can write $f_k = x^{i_k} y^{j_k} + \sum_{l=1}^{n} a_{kl} m_l$.

eg $m = \langle x^3, xy, y^2 \rangle$



$$f_0 = \overset{y}{x^3} + \cancel{a_{01}} + \overset{y}{a_{02}}x + \overset{y}{a_{03}}x^2 + \overset{y}{a_{04}}y$$

$$f_1 = \overset{x^2}{xy} + \cancel{a_{11}} + \overset{x^2}{a_{12}x} + \overset{x^2}{a_{13}}x^2 + \overset{x^2}{a_{14}}y$$

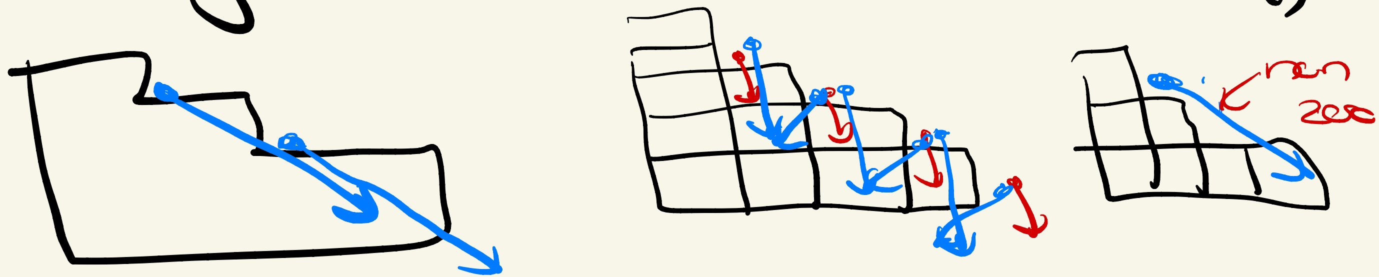
$$f_2 = y^2 + \cancel{a_{21}} + \cancel{a_{22}x} + a_{23}x^2 + a_{24}y$$

We need $yf_0 = x^2 f_1$, $yf_1 = xf_2$ in S/m

$a_{01} = a_{11} = 0 = a_{21} = a_{22} = 0$

So $\dim_k \mathrm{Hom}(m, S/m) = 8 = 2N$

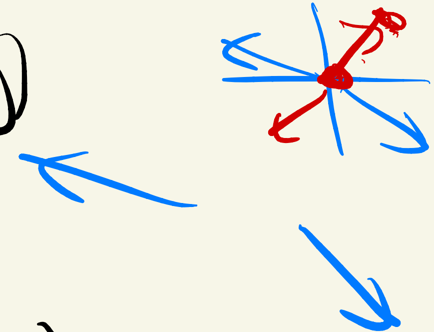
We represent a_{kl} as an arrow from $x^{i_k}y^{j_k}$ to m_l & write (k, m_l)



Our conditions on syzygies say that if an arrow can be moved down & to the right (or up & left) keeping the tail on the boundary & the head beneath the staircase, the coeffs a_{kl} agree. If the head crosses an axis, $a_{kl} = 0$.

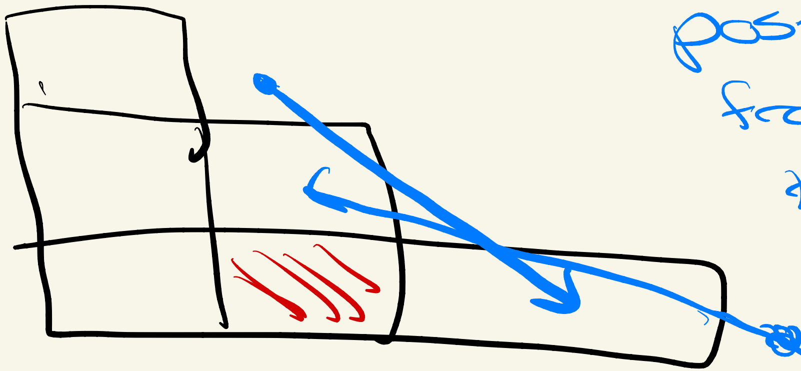
Nonzero arrows point (weakly) northwest or southeast

negative positive

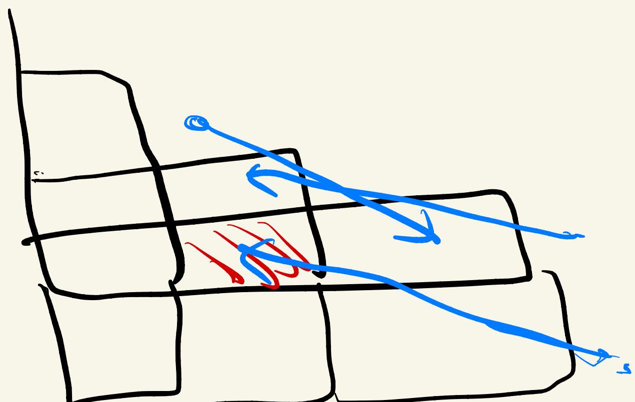


We now show there are 2N equivalence classes of nonarrows, subject to our rules from the previous slide..

To each box in the Young diagram we associate 2 arrows



positive:
from top of column to almost end of row (then move left to hit a min generator)



negative
from end of row to almost end of column (then down to hit a minimal generator)

Claim: Every equivalence class of arrows shows up here

(move a positive arrow down/right as far as possible, and then further right until the head is the last box in its row)

$\Rightarrow$ # equiv. classes is $\leq 2N$

$\Rightarrow \dim_k (m, S/m) \overset{\leq}{=} 2N$

$\Rightarrow \mathrm{Hilb}^N(\mathbb{A}^2)$ is smooth at $[m]$

Miller - Sturmfels Comb. Commut. alg