TCC Hilbert schemes and Moduli spaces

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Last time: Construction of $\mathrm{Hilb}_P(\mathbb{P}^n)$ as a subscheme of a Grassmannian.

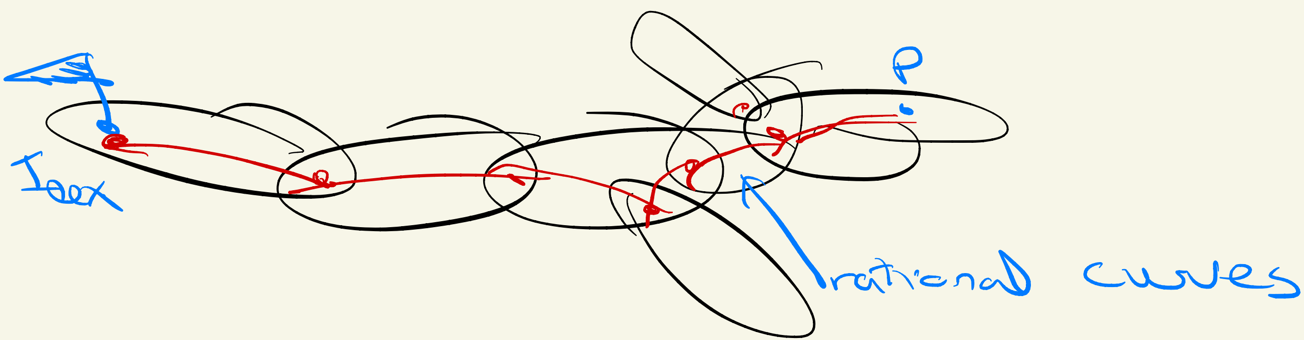
Today Connectedness of $\mathrm{Hilb}_P(\mathbb{P}^n)$ and Hilbert schemes $\mathrm{Hilb}(X)$ of other varieties/schemes.

Connectedness

Theorem (Hartshorne)

The Hilbert scheme $\mathrm{Hilb}_p(\mathbb{P}^n)$ is connected.

We'll actually show it is rationally chain connected



Weird fact: The proof does not use the fact that $\mathrm{Hilb}_p(\mathbb{P}^n)$ exists (or details of construction)

Key construction Gröbner degeneration.

Given a homogeneous ideal $I \subseteq K[x_0, \dots, x_n]$ and a weight vector $w \in \mathbb{N}^n$ we construct the ideal

$$I_t = \langle \tilde{f} : f \in I \rangle \subseteq K[x_0, \dots, x_n, t]$$

where for $f = \sum c_u x^u$ we have

$$\tilde{f} = t^{\max(w \cdot u)} \sum c_u t^{-w \cdot u} x^u = t^{\max(w \cdot u)} f\left(\frac{x_i}{t^{w_i}}\right)$$

$\tilde{f} = t^{\max(w\cdot u)} f\left(\frac{x_i}{t^{w_i}}\right)$ $\quad I_t = \langle \tilde{f} : f \in I \rangle \subseteq K[x_0, , x_n, t]$

eg Let $I = \langle x_0x_2 - x_1^2 \rangle \subseteq K[x_0, x_1, x_2]$.

For $w = (10, 5, 1)$ $f = x_0x_2 - x_1^2$ we have

$$\tilde{f} = t^{11}\left(t^{-11}x_0x_2 - t^{-10}x_1^2\right) = x_0x_2 - tx_1^2$$

$$\tilde{I} = \langle x_0x_2 - tx_1^2 \rangle$$

↯ $\tilde{I}$ is not always generated by $\{\tilde{f}_i : f_i \text{ generate } I\}$ but true for principal ideals

For $w = (1, 5, 1)$,

$$\tilde{f} = t^{10}\left(t^{-2}x_0x_2 - t^{10}x_1^2\right) = t^{8}x_0x_2 - x_1^2$$

The ideal $\tilde{I} = \langle \tilde{f} : f \in I \rangle \subseteq K[x_0, \dots, x_n, t]$ defines a subscheme of $\mathbb{P}^n \times \mathbb{A}^1$, and the inclusion $K[t] \to \dfrac{K[x_0, \dots, x_n, t]}{\tilde{I}}$ induces a morphism

$$\begin{array}{c} \operatorname{Proj}\left(\dfrac{K[x_0, \dots, x_n, t]}{\tilde{I}} \right) \\ \pi \big\downarrow \\ \mathbb{A}^1. \end{array}$$

Key fact: π is flat, and all fibres over $t \neq c$ are isomorphic to $\operatorname{Proj}\left(\dfrac{K[x_0, \dots x_n]}{I}\right)$. The fibre over 0 is defined by the initial ideal in the sense of Gröbner bases eg $I = \langle x_0 x_2 - x_1^2 \rangle$ ← Veronese embedding of $\mathbb{P}^1$ into $\mathbb{P}^2$

$w = (10, 5, 1)$ $\quad \tilde{I} = \langle x_0 x_2 - t x_1^2 \rangle$

Fibre over 0: $\langle x_0 x_2 \rangle$ ← 2 coordinate lines

$w = (1, 5, 1)$ $\quad \tilde{I} = \langle t^8 x_0 x_2 - x_1^2 \rangle$

Fibre over 0: $\langle x_1^2 \rangle$ ⟵ double line.

Flat: $\dfrac{K[x_0, x_1, x_2, t]}{\langle x_0 x_2 - t x_1^2 \rangle}$ is a free $K[t]$ module with basis {monomials in x_0, x_1, x_2 not divisible by $x_0 x_2$}

Gröbner references:

Basics: Cox, Little, O'Shea. Ideals, Varieties, and algorithms

Gröbner degenerations: Eisenbud Commutative algebra §15.8

Sturmfels Gröbner bases & convex polytopes

Connectedness Proof

Step 1: Reduce to the case that I is Borel-fixed.

The group $GL(n+1,K)$ acts on $K[x_0,\dots,x_n]$ by linear change of coordinates:

$$x_i \longmapsto \sum_{j=0}^{n} a_{ji} x_j$$

eg
$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot (x_0^2 + 2x_0x_1 + x_1^2) = (x_0+3x_1)^2 + 2(x_0+3x_1)(2x_0+4x_1) + (2x_0+4x_1)^2$$
$$= 9x_0^2 + 42x_0x_1 + 49x_1^2$$

$$x_i \mapsto \sum_{j=0}^{n} a_{ji} x_j$$

We consider the action of the Borel group of upper triangular matrices

Lemma When char $K = 0$, an ideal $I \subseteq K[x_1, x_n]$ is fixed by the action of the Borel group if & only if

1) I is a monomial ideal

2) For all monomials $x^u \in I$ ($x_1^{u_1} \cdots x_n^{u_n}$) & $x_i \mid x^u$, we have $\frac{x_j}{x_i} x^u \in I$ for all $j < i$.

Strongly stable

Borel-fixed ideals are strongly stable monomial ideals.

Sketch for monomial

That I is monomial follows from the fact that it is fixed by T = diagonal matrices $\subseteq$ upper triangular matrices

T acts by scaling variables:

$$(t_1, \ldots, t_n) \cdot x_i = t_i x_i.$$

To see T-fixed $\Rightarrow$ monomial:

If $x_0 + x_1 \in I \subseteq K[x_0, x_1]$, then $(1,t) \cdot (x_0 + x_1)$

$= x_0 + t x_1 \in I$ for all t, so $(x_0 + x_1) - (x_0 + t x_1) = (1-t) x_1 \in I \Rightarrow x_1 \in I$

$x_0 \in I$

eg $I = \langle x_0^2, x_0x_1, x_1^2 \rangle \subseteq K[x_0, x_1, x_2]$ is Borel-fixed while $J = \langle x_0^2, x_1^2, x_2^2 \rangle$ is not.

$$x_1^2 \in J \quad \text{but} \quad \frac{x_0}{x_1} x_1^2 = x_0x_1 \notin J.$$

Useful Facts

1) The saturation of a Borel-fixed ideal is Borel-fixed

2) There are only finitely many saturated Borel-fixed ideals with a given Hilbert polynomial.

Proposition Fix a homogeneous ideal $I \subseteq K[x_0, \dots, x_n]$, and generic $\underline{w}$. There is an open set $U \subseteq GL(n+1, K)$ for which $in_w(gI)$ is constant for $g \in U$. This is Borel-fixed for $w_0 > w_1 > \dots > w_n$.

generic initial ideal $gin_w(I)$.

eg $I = \langle x_1 + x_2 \rangle \subseteq K[x_0, x_1, x_2]$. For $g = (g_{ij})$

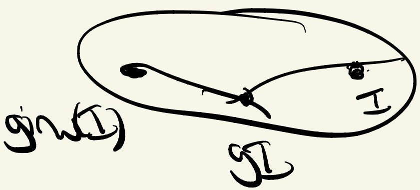
$$gI = \langle g_{01}x_0 + g_{11}x_1 + g_{21}x_2 + g_{02}x_0 + g_{12}x_1 + g_{22}x_2 \rangle$$

$$= \langle (g_{01} + g_{02})x_0 + (g_{11} + g_{12})x_1 + (g_{21} + g_{22})x_2 \rangle$$

For $w = (10, 5, 1)$ & $U = \{g_{01} + g_{02} \neq 0\} \subseteq GL(3, K)$

$in_w(gI) = \langle x_0 \rangle$. ← Borel fixed

Thus starting at $[I] \in \mathrm{Hilb}_P(\mathbb{P}^n)$, we choose a path in $GL(n+1, K)$ to a pt g in $\mathcal{U}$, which gives a path $[I] \rightsquigarrow [gI]$, then take the initial ideal $[gI] \rightsquigarrow [\mathrm{gin}([I])] \leftarrow$ Borel-fixed.



Step 2: Move towards the lexicographic ideal

Given a Borel-fixed ideal I we construct an ideal J with exactly two initial ideals:

$in_w(J) = I$, $in_{-w}(J) = I'$,

really $N(1,\dots,1) - w$ for $N \gg 0$

where I' is closer to the lexicographic ideal. We then take $gin_{lex}(I')$ to get another Borel-fixed ideal closer to I_{lex} than I.

only finitely many, so must terminate

This is the approach of Peeva & Stillman

Idea of construction

Let I be a saturated Borel-fixed ideal not equal to the lexicographic ideal.

Let d be the smallest degree in which I_d is not a lex-segment ← spanned by $\binom{n+d}{n} - P(d)$ largest monomials in lex order

Let m be the largest monomial of degree d not in I, and let f be the largest monomial smaller than m with $f \in I$. f will be a minimal generator

First approximation: $J = \langle f - m, \text{ other gens of } I \rangle$

(actually need to modify to add other binomials

eg if $f - m$ is $x_1^2 - x_0 x_1$ & $x_1 x_2 \in I$, add $x_1 x_2 - x_0 x_2$

eg $P(t) = 4$ $n = 2$.

$$I = \langle x_0^2, x_0x_1, x_1^3 \rangle$$

$$I' = \langle x_0, x_1^4 \rangle$$ ← lexicographic ideal

are the only Borel-fixed saturated ideals in $K[x_0, x_1, x_2]$ with Hilbert polynomial P.

$$J = \langle x_0^2, x_0x_1, x_1^3 - x_0x_2^2 \rangle$$

$$\mathrm{in}_{(1,10,1)}(J) = I$$

$$\mathrm{in}_{(10,1,10)}(J) = \langle x_0^2, x_0x_1, x_0x_2^2, x_1^4 \rangle$$

which has saturation $\langle x_0, x_1^4 \rangle = I'$.

Other Hilbert Schemes

Let X be a projective scheme.

$$\mathrm{Hilb}(X)(B) = \left\{ \begin{array}{l} \text{closed subschemes } Z \subseteq X \times B \\ \text{with } \begin{array}{c} Z \\ \downarrow \\ B \end{array} \text{ flat \& proper} \end{array} \right\}$$

To show that this is representable, we fix an embedding $X \subseteq \mathbb{P}^N$, which defines a Hilbert polynomial for subschemes of X

$$\mathrm{Hilb}_P(X)(B) = \left\{ \begin{array}{l} \text{closed subschemes } Z \subseteq X \times B \subseteq \mathbb{P}^n_B \\ \text{flat over } B, \text{ where all fibres have} \\ \text{Hilbert polynomial } P. \end{array} \right\}$$

We get $\mathrm{Hilb}_P(X)$ as a closed subscheme of $\mathrm{Hilb}_P(\mathbb{P}^N)$

Lemma The locus $Z \subseteq Gr(r,n)$ of r-dim subspaces of K^n containing a fixed vector $v \in K^n$ is closed.

Pf Given $V \in Gr(r,n)$, pick a basis for V, and write this as the rows of an $r \times n$ matrix

$$\begin{array}{cc} & n \\ r & \begin{pmatrix} \text{———} \\ \text{———} \\ \text{———} \\ \text{———} \end{pmatrix} \end{array} \leftarrow \text{rank } r$$

If $v \in V$, the rank of $\begin{array}{cc} & n \\ r+1 & \begin{pmatrix} \text{—}\ \underline{v}\ \text{—} \\ \text{———} \\ \text{———} \\ \text{———} \end{pmatrix} \end{array}$ is r, so all $(r+1)\times(r+1)$ minors vanish. Expand these along the first row - they have the form $\sum_j \pm v_j P_{I_j}$

$\uparrow$ linear in P_{I_j} for fixed $\underline{v}$.

So Z is the intersection of $Gr(r,n)$ in its Plücker embedding with a subspace of $\mathbb{P}^{\binom{n}{r}-1}$.

Construction of $\mathrm{Hilb}_P(X)$

Fix D at least the Gotzmann number of P and the degrees of generators of the ideal I_X of X. Fix a basis $f_1, \ldots, f_r$ of $(I_X)_D$. We want $I_X \subseteq I$, so $f_i \in I_D$ for $1 \leq i \leq r$.

situation of lemma.

This describes $\mathrm{Hilb}_P(X)$ as a closed subscheme of $\mathrm{Hilb}_P(\mathbb{P}^N)$.

Question Does this depend on the choice of embedding of X into $\mathbb{P}^N$?

Answer The decomposition into pieces indexed by Hilbert polynomials might, but not $\mathrm{Hilb}(X)$. $\leftarrow$ magic of Yoneda's lemma!!

↯ $\mathrm{Hilb}_P(\mathbb{P}^n)$ is always connected, but $\mathrm{Hilb}_P(X)$ might not be, even if X is connected.

eg $X = V(f) \subseteq \mathbb{P}^3$ smooth cubic surface.

$\mathrm{Hilb}_{t+1}(X)$ is 27 points

eg $X = V(f) \subseteq \mathbb{P}^3$ smooth conic.

$$\mathrm{Hilb}_{t+1}(X) = \mathbb{P}^1 \amalg \mathbb{P}^1$$

2 rulings