TCC - HILBERT SCHEMES AND MODULI SPACES - LECTURE 8

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0.1. **Clarification from last lecture.** When we grade the polynomial ring $S = K[x_1, \dots, x_n]$ by an abelian group, S_aS_b means the vector space *spanned* by the products $\{fg : f \in S_a, g \in S_b\}$. This is analogous to the product ideal IJ, which is the ideal *generated* by the products $\{fg : f \in I, g \in J\}$.

0.2. **The moduli space of curves.** Last time we saw the moduli functor for the moduli space M_g of smooth curves of genus g:

$$B \mapsto \{ \text{ isomorphism classes of flat proper families } \pi\colon C \to B \text{ whose geometric fibers are smooth complete connected curves of genus } g\},$$

where $C \to B$ is isomorphic to $C' \to B$ if there is an isomorphism $\phi\colon C \to C'$ with

$$\begin{tikzcd} C \arrow[rr, "\phi"] \arrow[dr] & & C' \arrow[dl] \\ & B & \end{tikzcd}$$

We saw that this functor is not always representable. Today we will discuss its coarse moduli space, generalisations, and the Deligne-Mumford compactification.

We first expand to the moduli space $M_{g,n}$ of genus g curves with n distinct marked points. This has the moduli functor

$$B \mapsto \{ \text{ isomorphism classes of flat proper families } \pi\colon C \to B \text{ whose geometric fibers are smooth complete connected curves of genus } g \text{ together with } n \text{ non-intersecting sections } \sigma_i\colon B \to C \text{ for } 1 \leq i \leq n\},$$

where a pair $(\pi : C \to B, \sigma_1, \dots, \sigma_n)$ is isomorphic to $(\pi' : C' \to B, \sigma'_1, \dots, \sigma'_n)$ if there is an isomorphism $\phi\colon C \to C'$ that makes $\pi\colon C \to B$ isomorphic to $\pi'\colon C' \to B$ with $\sigma'_i = \phi \circ \sigma_i$ for $1 \leq i \leq n$.
Warning: For an algebraic geometer, "marked" means labelled (so point one is distinguishable from point two). Topologists also consider

the space $M_{g,n}$, but do not label the points. This means that their version of $M_{g,n}$ is (essentially) our $M_{g,n}/S_n$.

This can lead to confusion. For example, $M_{0,n}$ is a smooth variety, but a topologist thinks that is an orbifold, with singularities coming from symmetric placements of the points.

0.3. **Genus 0.** The moduli functor for $M_{0,n}$ is representable by a smooth variety when $n \geq 3$. This is easy to describe. The only smooth curve of genus zero is $\mathbb{P}^1$. The key fact we use is that the automorphism group PGL(2) of $\mathbb{P}^1$ is uniquely three transitive: there is a unique automorphism taking any three distinct ordered points on $\mathbb{P}^1$ to any other three distinct ordered points. (Exercise!) This means that given $(\mathbb{P}^1, p_1, \dots, p_n)$, we can take p_1 to $[1:0]$, p_2 to $[1:1]$, and p_3 to $[0:1]$ ("0, 1, and ∞"), and just record the locus $p_4, \dots, p_n$. These are distinct points in $\mathbb{P}^1$ that are not equal to $0, 1,$ or ∞. Thus $M_{0,3}$ is a point, $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$, and

$$\begin{aligned} M_{0,n} &= (\mathbb{P}^1 \setminus \{0,1,\infty\})^{n-3} \setminus \text{ diagonals} \\ &= (\mathbb{C}^*)^{n-3} \setminus \{x_i = 1, x_i = x_j\} \\ &= \mathbb{P}^{n-3} \setminus \{x_i = 0, x_i = x_j\}. \end{aligned}$$

The first equality here comes from writing $p_j = [1 : x_{j-4}]$ for $4 \leq j \leq n$. This exhibits $M_{0,n}$ as a the complement of a hyperplane arrangement in $\mathbb{P}^n$, and thus as a smooth variety. It has dimension $n-3$.

0.4. **Genus 1.** We consider $M_{1,1}$. Every smooth genus one curve with one marked point p is isomorphic to $V(y^2z - f_3(x,z))$, where $f_3 = (x - \lambda_1 z)(x - \lambda_2 z)(x - \lambda_3 z)$, and $p = [0:1:0]$. Smoothness implies that the λ_i are distinct. Linear changes of coordinates in x and z give isomorphisms of the curve, so we may assume that f_3 is $x(x-1)(x-\lambda)$ for $\lambda \neq 0, 1$. This is using the three-transitivity for $\mathrm{Aut}(\mathbb{P}^1)$ mentioned above. Note, however, that there is no intrinsic order on $\lambda_1, \lambda_2, \lambda_3$, so there are several options for λ, depending on which of $\lambda_1, \lambda_2, \lambda_3$ is taken to $0, 1$. These are the 6 different choices for the *cross-ratios* of $\lambda_1, \lambda_2, \lambda_3$: $\lambda, 1-\lambda, 1/\lambda, 1/(1-\lambda), (\lambda-1)/\lambda, \lambda/(\lambda-1)$. The map

$$\lambda \mapsto \frac{256(1 - \lambda(1-\lambda))^3}{(\lambda(1-\lambda))^2} \tag{1}$$

is invariant under these choices. It has image $\mathbb{A}^1$, which is a coarse moduli space for $M_{1,1}$. The invariant (1) is called the j-invariant of the curve, and $M_{1,1}$ is the j-line. It has dimension 1.

0.5. **Genus 2.** Every curve of genus 2 is hyperelliptic: it has a degree 2 map $\phi\colon C \to \mathbb{P}^1$, given by the two sections of the canonical divisor. This morphism has 6 ramification points by the Riemann-Hurwitz formula, and the curve can be reconstructed from these: on an affine chart it is $y^2 = \prod_{i=1}^{6}(x-\lambda_i)$. The $\lambda_1, \dots, \lambda_6$ are six (unordered) points in $\mathbb{P}^1$, so give a point of $M_{0,6}/S_6$. Descriptions of the corresponding invariants were given originally by Igusa. This means that $\dim(M_2) = \dim(M_{0,6}) = 3$.

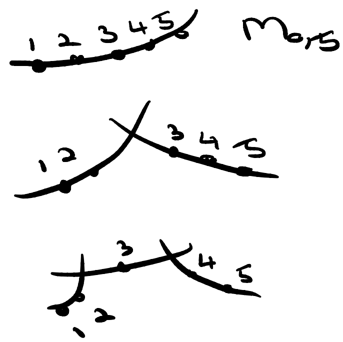
0.6. **General** g**.** In general $\dim(M_{g,n}) = 3g-3+n$. Low genus cases are deceptive, however. The moduli space M_g, is not rational (birational to $\mathbb{P}^n$ for some n) or even unirational (admits a dominant map $\mathbb{P}^N \dashrightarrow M_g$) for high g. In fact M_g has general type (the canonical divisor K_{M_g} is big and effective) for $g \geq 22$. Recall that Kodaira dimension (the dimension of the canonical model of a variety) is a birational invariant, so a variety cannot be simultaneously rational (Kodaira dimension zero) and general type (Kodaira dimension equal to the dimension of the variety).

0.7. **Compactifications.** The moduli spaces M_g and $M_{g,n}$ are not compact. This is easy to see: $M_{0,4} = \mathbb{P}^1 \setminus \{0, 1, 2\}$. We now describe the most popular compactification: the Deligne-Mumford compactification of $M_{g,n}$, which is the moduli space of *stable* curves of genus g with n marked points.

Definition 1. A curve C has a *node* at a point p, also called an ordinary double point, if the completed local ring $\widehat{\mathcal{O}}_{C,p}$ is isomorphic to $K[[x,y]]/\langle xy\rangle$. When we work over $\mathbb{C}$ this means that it has a neighbourhood in the analytic topology looking like $xy = 0$. The curve C is *nodal* if every point is smooth or a node.

When talking about the genus of a non-smooth curve, we could mean either the *geometric* genus or the *arithmetic* genus. The geometric genus is the genus of the normalization, which is smooth. For example, if two irreducible components intersect in a node, the normalization is just the two components, so the geometric genus is the sum of the genuses of the components. However the geometric genus is not constant in flat families. For example consider $Y = V(xyz + t(x^3 + y^3 + z^3)) \subseteq \mathbb{P}^2 \times \mathbb{A}^1$, and $\pi\colon Y \to \mathbb{A}^1$. For $t \neq 0$ the fiber Y_t is a smooth plane cubic, so has genus one. The special fiber Y_0 is the union of three lines intersecting in nodes, so has geometric genus zero. The *arithmetic* genus, by contrast, is invariant in flat families over nice bases. When $C \subset \mathbb{P}^n$ has Hilbert polynomial $P(t) = dt + e$, the arithmetic genus is $1 - e$. Note that this equals the geometric genus when C is smooth by Riemann Roch.

FIGURE 1. The stable genus zero curves for $n = 5$



The arithmetic genus is defined to be $\dim H^1(C, \mathcal{O}_C)$. The genus of a connected nodal curve is the genus of the *dual graph* of C plus the genuses of each component. This is the number of nodes, minus the number of irreducible components, plus one, plus the sum of the genuses of the components.

Definition 2. A connected nodal curve C with n marked points $p_1, \dots, p_n$ is *stable* if the marked points are smooth points of C, and every genus zero component has at least three special points (nodes or marked points), and every genus one component has at least one special point.

Stability of a curve implies that the automorphism group is finite. It also implies that the dualizing sheaf ω_C is ample.

Example 3.
(1) Stable curves of arithmetic genus 0 are "trees of $\mathbb{P}^1$"s, with at least three nodes or marked points on each component. Figure 1 shows the options for $n = 5$, and Figure 2 shows the options for $n = 6$.
(2) Figure 3 shows the options for stable curves of (arithmetic) genus two. The red numbers indicate the genus of each curve. The green cartoons are the topological pictures in each case.

Definition 4. The moduli space of *stable* curves of genus g with n marked points has moduli functor

$$B \mapsto \{ \text{ isomorphism classes of flat proper families } \pi\colon C \to B \text{ whose} \\ \text{geometric fibers are stable complete connected curves of arithmetic genus } g \\ \text{together with } n \text{ non-intersecting sections } \sigma_i\colon B \to C \text{ with images in the smooth} \\ \text{loci of fibers for } 1 \le i \le n\}.$$

FIGURE 2. The stable genus curves for $n = 6$

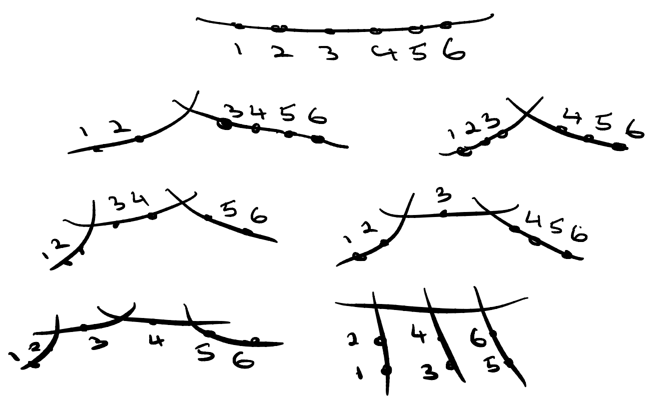
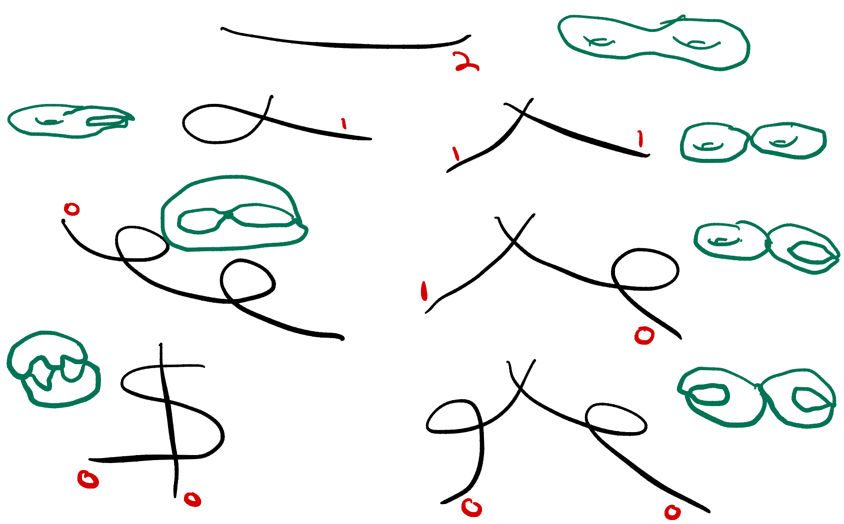


FIGURE 3. The stable curves for genus 2



Theorem 5. *There is a projective coarse moduli space $\overline{M}_{g,n}$ parameterizing stable curves of arithmetic genus g with n marked points.*

This is due to Deligne and Mumford [DM69], who actually introduce the moduli stack. Projectivity is due to Knudsen [KM76, Knu83a, Knu83b]. For $g \geq 2$ Mumford constructs this coarse moduli space as a GIT quotient of a locus in a Hilbert scheme of curves; this one of the main motivations for the development of geometric invariant theory (GIT). The genus zero case is simpler; in this case $\overline{M}_{0,n}$ is a *fine* moduli space, originally introduced by Grothendieck, and we have several different explicit descriptions of it, including as blow-up of $\mathbb{P}^{n-3}$.

We have not begun to scratch the surface of this topic. Some further references are [Cav16], [HM98], [KV07].

References

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